A Combinatorial Proof of a Fixed Point Property

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Abstract

A class of finite simplicial complexes, called *pseudo cones*, is developed that has a number of useful combinatorial properties. A partially ordered set is a pseudo cone if its order complex is a pseudo cone. Pseudo cones can be constructed from other pseudo cones in a number of ways. Pseudo cone ordered sets include finite dismantlable ordered sets and finite truncated noncomplemented lattices. The main result of the paper is a combinatorial proof of the fixed simplex property for finite pseudo cones in which a combinatorial structure is constructed that relates fixed simplices to one another. This gives combinatorial proofs of some well known non-constructive results in the fixed point theory of finite partially ordered sets.

Keywords: fixed point property; dismantlable ordered set; lattice; noncomplemented lattice; straightening law. *MSC2010:* 55M20; 05E45; 06A07

1 Introduction

One of the open problems [6, 9] in the fixed point theory of partially ordered sets is to find a combinatorial proof that every finite truncated noncomplemented lattice has the fixed point property. The original proof [5, Theorem 2.1] used algebraic methods. The result was therefore not combinatorial. The purpose of this paper is to provide a combinatorial proof of this result as well as of a number of related results in [2, 3, 5, 6].

The basic structure we develop as the basis for the combinatorial proof is called a *pseudo cone* (PC), defined in Section 3. A pseudo cone is a generalization of the notion of a cone with a distinguished peak vertex. A PC structure is essentially the same as an acyclic complete matching from discrete Morse theory

[14]. Consequently, the geometric realization of a PC is contractible and hence homologically acyclic.

One of the important advantages of algebraic properties like acyclicity is that one can construct larger acyclic simplicial complexes by "patching" together or extending smaller acyclic complexes. This is basically how dismantlable ordered sets and truncated noncomplemented lattices were originally shown to be acyclic. We develop an extension technique that applies to pseudo cones in Section 5, and this technique is then used to show that dismantlable ordered sets and truncated noncomplemented lattices are pseudo cones in Sections 6 and 7, respectively.

The disadvantage of an algebraic property like acyclicity is that the fixed point property is an abstract existence result. The fixed point is shown to exist by exhibiting a property of the fixed point set that excludes the possibility that the fixed point set is empty. Pseudo cone structures remedy this defect of algebraic methods. In this case, a nonempty combinatorial structure is constructed that relates fixed points to one another. The combinatorial proof of the fixed point property is given in Section 8. The proof makes use of a generalization of a fixed simplex, called a *hit*, which is the basic unit of the combinatorial structure that is constructed.

For simplicity in the sequel, all sets are assumed to be finite.

2 Combinatorial Background

We assume as background the notions of *multisets*, *partially ordered sets* (or *posets*, for short) and *simplicial complexes*. As stated above, we assume that all posets and simplicial complexes are finite.

2.1 Multisets

A *multiset* (or *bag*), is a generalization of a set in which elements can occur more than once. One can formalize a multiset as a function $m\colon U \to \mathbb{N}$, where U is the universe from which elements may be chosen. For a multiset m, the *multiplicity* of $x \in U$ will be written in functional notation as $m(x)$. It is sometimes convenient to view multisets as if they were sets. For example, we will write $y \in m$ to mean that $m(y) \geq 1$, and we will use the set-builder notation to construct multisets. The analogue of the union of sets is the sum of multisets, and we will use addition and summation to indicate this operation to avoid confusion with the set-theoretical concepts. Unlike the case of sets, one can multiply a multiset by a nonnegative integer.

2.2 Partially Ordered Sets

Let P be a poset with order relation $\leq$. The inverse relation of $\leq$ is also a partial order, called the *dual* order. A subset $V \subseteq P$ is called an *(order) filter* if it is closed with respect to $\geq$, i.e., if $x \in V$ and $y \geq x$, then $y \in V$. The *principal filter* of an element $x \in P$ is the smallest filter containing x. The principal filter is written $V(x)$ and is the set $\{y \in P \mid y \geq x\}$. Dually, a subset of P is an *(order) ideal* if it is closed with respect to $\leq$, and the *principal ideal* of $x \in P$ is written $J(x) = \{y \in P \mid y \leq x\}$. An element $x \in P$ is said to *cover* $y \in P$ if y is a maximal element in $J(x) - \{x\}$. This relation is written $y \lessdot x$. If we wish to allow y to be the same as x as well as be covered by x, then we write $y \,\underline{\lessdot}\, x$.

Let P and Q be two posets. A function $f \colon P \to Q$ is said to be *order-preserving* if $f(x) \leq f(y)$ in Q whenever $x \leq y$ in P. If $P = Q$, then f is said to be a *self-map*. A *fixed point* of a self-map is an element $x \in P$ such that $f(x) = x$. A poset P is said to have the *fixed point property* if every order-preserving self-map has at least one fixed point.

A subset $S \subseteq P$ of a poset is always implicitly endowed with the order relation obtained by restricting the order relation of P to S. A *chain* of a poset P is a subset $C \subseteq P$ such that C is totally ordered as a poset.

Let P and Q be two posets. The *cardinal power* Q^P is the set

$$\{f \colon P \to Q \mid f \text{ is order-preserving}\},$$

where functions are partially ordered pointwise (i.e., $f \leq g$ if and only if for every $x \in P$, $f(x) \leq g(x)$). In particular, the poset of order-preserving self-maps of P is P^P.

A *lattice* L is a poset such that every pair of elements $x, y \in L$ has a least upper bound (join) $x \vee y$, and a greatest lower bound (meet) $x \wedge y$. If a lattice has a maximum element, we write it $\hat{1}$, and if it has a minimum element, we write it $\hat{0}$. The *truncation* or *proper part* of a lattice is $\breve{L} = L - \{\hat{0}, \hat{1}\}$. Two elements $x, y \in \breve{L}$ are said to be *lower semi-complements* if $x \wedge y = \hat{0}$. One similarly defines *upper semi-complements*. Two elements $x, y \in \breve{L}$ are said to be *complements* if they are both upper and lower semi-complements. A lattice is said to be *complemented* if every element has a complement, and the lattice is *noncomplemented* otherwise. (See [7] for more about lattices.)

2.3 Simplicial Complexes

For a set S, let $F(S)$ be the poset of finite subsets of S. A *(reduced) simplicial complex* Σ is a nonempty order ideal of $F(S)$ for some set S. The *vertex set* of a simplicial complex Σ, is $\{s \in S \mid \{s\} \in \Sigma\}$. It is easy to see that a simplicial complex Σ with vertex set V is an order ideal of $F(V)$. The elements of V are called the *vertices* of Σ, while an element $\sigma \in \Sigma$ is called a *simplex* of Σ. To simplify the presentation of examples, we will use one-digit integers as labels for vertices and a simplex will be written by concatenating the vertex labels in order. Thus the simplex $\{1, 3, 5\}$ will be abbreviated 135. The *length* (cardinality) of a simplex $\sigma \in \Sigma$ will be written $|\sigma|$. The *length* of a simplicial complex is the maximum length of any of its simplices. If $W \subseteq V$, then the *subcomplex* of Σ *restricted* to W is the simplicial complex $\Sigma|W = \{\sigma \in \Sigma \mid \sigma \subseteq W\}$. We say that W is *connected* if for every $v, w \in W$, there exists a sequence $v = w_0, w_1, \ldots, w_n = w$ of vertices such that for $i = 1, \ldots, n$, $\{w_{i-1}, w_i\} \in \Sigma|W$.

Let $\sigma \in \Sigma$. The *(closed) star* of σ in Σ is the simplicial complex $St_\Sigma(\sigma) = \{\tau \in \Sigma \mid \tau \cup \sigma \in \Sigma\}$. In particular, $St_\Sigma(\emptyset) = \Sigma$. The *link* of σ in Σ is the simplicial complex $Lk_\Sigma(\sigma) = \{\tau \in \Sigma \mid \tau \cup \sigma \in \Sigma \text{ and } \tau \cap \sigma = \emptyset\}$. When $\sigma = \{v\}$, we will write $St_\Sigma(v)$ for $St_\Sigma(\{v\})$, and $Lk_\Sigma(v)$ for $Lk_\Sigma(\{v\})$. When there exists a vertex v such that $\Sigma = St_\Sigma(v)$, then we say that Σ is a *cone* (with *peak* v).

For simplicial complexes Σ on V and Σ' on V', a *simplicial map* is a function from vertices to vertices, $f\colon V \to V'$, such that for every simplex $\sigma \in \Sigma$, we have that $f(\sigma) \in \Sigma'$. A simplicial map $f\colon V \to V'$ induces an order-preserving map $P(f)\colon \Sigma \to \Sigma'$ in the obvious way. As a result, there are two possible interpretations of the "fixed point property" for a simplicial complex. One can use arbitrary order-preserving maps on the simplicial complex, or one can restrict attention to the simplicial maps. That these two interpretations are equivalent is shown in the following:

Proposition 2.1 *For a finite simplicial complex Σ on vertex set V, the following are equivalent:*

1. *For every simplicial map $f\colon V \to V$, there exists a nonempty simplex $\sigma \in \Sigma$ such that $f(\sigma) = \sigma$.*
2. *For every order-preserving map $f\colon \Sigma \to \Sigma$ there exists a nonempty simplex $\sigma \in \Sigma$ such that $f(\sigma) = \sigma$.*

Proof The implication (2)$\Rightarrow$(1) is trivial. To show (1)$\Rightarrow$(2), let $f\colon \Sigma \to \Sigma$ be an order-preserving map. By finiteness, there exists a function $\phi\colon \Sigma \to V$ such that for every nonempty $\sigma \in \Sigma$, we have that $\phi(\sigma) \in \sigma$. Define a function

$g\colon V \to V$ such that for every $v \in V, g(v) = \phi(f(\{v\}))$. To show that g is a simplicial map, let $\sigma \in \Sigma$ and $v \in \sigma$. By definition of g and ϕ, we have that $g(v) = \phi(f(\{v\})) \in f(\{v\})$. Since f is order-preserving, $f(\{v\}) \subseteq f(\sigma)$. Therefore $g(v) \in f(\sigma)$. This is true for every $v \in \sigma$, so $g(\sigma) \subseteq f(\sigma)$. It follows that $g(\sigma) \in \Sigma$, and g is a simplicial map. By hypothesis (1), there exists a nonempty simplex $\tau \in \Sigma$ such that $g(\tau) = \tau$. We have already shown that $g(\sigma) \subseteq f(\sigma)$ in general, so in particular we have that $\tau = g(\tau) \subseteq f(\tau)$. Since Σ is finite, the ascending sequence $\tau \subseteq f(\tau) \subseteq f(f(\tau)) \subseteq \ldots$ must terminate in a finite number of steps to a nonempty fixed simplex. ■

When one of the two equivalent conditions in Proposition 2.1 holds, we say that the simplicial complex has the *fixed simplex property.* The reason we did not call this the fixed *point* property is that one might reasonably interpret "point" to mean "vertex," and this property is definitely not a fixed vertex property. However, in the next section we show that for posets, a fixed simplex property implies the fixed point property.

2.4 The Order Complex

We now discuss the relationship between posets and simplicial complexes. Of course, a simplicial complex is a special kind of poset. In the other direction, the *order complex* of a poset P is the simplicial complex $\Delta(P)$ whose simplices are the chains of P, including the empty chain. The order complex is a special case of the *clique complex* of a graph, which consists of all complete subgraphs of the graph. The order complex of a poset P is the clique complex of the graph on P whose edges are the unordered pairs $\{x, y\}$ such that $x < y$ in P. If P is itself a simplicial complex, then $\Delta(P - \{\emptyset\})$ is called the *barycentric subdivision* of P. One can transfer simplicial complex notions to posets by using Δ. For example, a poset P is *connected* if and only if $\Delta(P)$ is connected. A poset P will be called a *cone* if $\Delta(P)$ is a cone, and a peak of $\Delta(P)$ will be called a peak of P. An order-preserving map of posets defines a simplicial map on the corresponding order complexes.

Proposition 2.2 *Let P be a poset. If $\Delta(P)$ has the fixed simplex property, then P has the fixed point property. The converse is not true.*

Proof Let $f\colon P \to P$ be an order-preserving self-map. If the corresponding simplicial self-map on the order complex fixes a nonempty simplex, then it fixes every element in the simplex because a simplex is a chain. Consequently, P has the fixed point property.

It should not be surprising that the converse is not true. There are many more simplicial self-maps on $\Delta(P)$ than those that arise from order-preserving self-maps. For example, let P be the poset consisting of the vertices, edges and faces of the square pyramid, ordered by containment. It is shown in [5, Example 2.4] that P has the fixed point property. However, the order complex $\Delta(P)$ does not have the fixed simplex property. To see this, label the vertices of the pyramid with $A, 0, 1, 2, 3$, so that A is the "apex" and the base of the pyramid is the square 0123. The poset P has 18 elements: 5 vertices, 8 edges and 5 faces. Define a simplicial automorphism on $\Delta(P)$ as follows. Interchange A with the base 0123. Map each vertex i to the face with vertices A, i and $i+1$ (mod 4), and this face is mapped to $i+1$ (mod 4). Map each edge with vertices A and i to the edge with vertices i and $i+1$ (mod 4), and this edge is mapped to the edge with vertices A and $i+1$ (mod 4). Consequently, when one applies this automorphism twice, it rotates the pyramid by $90°$. It is straightforward to check that this automorphism has no fixed simplices, so $\Delta(P)$ does not have the fixed simplex property. This counter-example proves that the converse does not hold. ■

3 Pseudo Cones

The simplest kind of contractible poset is one with a maximum or minimum element, or more generally one with an element that is comparable with every other element. The corresponding property for a simplicial complex is that it be a cone. We propose to generalize the notion of a cone.

If a simplicial complex Γ is a cone with peak v, then one may partition Γ into two equal-sized subsets: the simplices containing v and the simplices that do not contain v. Adding or removing v defines bijections between these two subsets, and every simplex containing v covers exactly one simplex that does not contain v. Accordingly, we will consider simplicial complexes Σ that can be partitioned into two subsets such that there is a bijection between them. It is helpful to have a simple word for specifying that a simplex is in one of the two subsets. We will call them *upper* and *lower* simplices. Upper simplices will be analogous to the simplices of a cone that contain its peak v; while the lower simplices will be analogous to the ones that do not. The names are suggestive of the fact that each upper simplex covers a lower simplex. Unlike cones, it is possible for an upper simplex to cover more than one lower simplex, but we will require that one of these lower simplices be the one that is bijectively related to the upper simplex. We now make this precise.

Definition 3.1 *Let Σ be a simplicial complex. A* pseudo cone structure *on Σ consists of:*

1. *A partition of Σ into upper and lower simplices;*

2. *A partial order $\preceq$ on the simplices such that:*

   (a) *For every upper simplex σ, the set of simplices that it covers (with respect to the containment order) has a smallest simplex (with respect to $\preceq$), and this simplex is a lower simplex. This simplex will be denoted $\gamma(\sigma)$;*

   (b) *The map γ is a bijection from the set of upper simplices onto the set of lower simplices, so it has an inverse which will be denoted β.*

A simplicial complex Σ is said to be a *pseudo cone* if it has at least one PC structure. A poset P is said to be a *pseudo cone* if $\Delta(P)$ is a pseudo cone. We now show that a cone is a special case of a pseudo cone. A pseudo cone that is not a cone will be called a *proper* pseudo cone.

Proposition 3.2 *A cone is a pseudo cone.*

Proof Let Σ be a cone with peak v. Define a simplex to be an upper simplex if and only if it contains v. Define a partial order on the simplices by $\sigma \prec \tau$ if and only if σ is a lower simplex and τ is an upper simplex. It is easy to check that this defines a PC structure on Σ. ■

One usually draws a poset by joining elements with an edge if one element covers the other. The resulting graph is called the *Hasse diagram* of the poset. The bijection γ maps each upper simplex to one of the simplices that it covers. As a result, γ can be regarded as defining a *complete matching* (also called a *perfect matching*) of the Hasse diagram of the simplicial complex. The existence of a complete matching implies that the number of simplices of a PC is always even. In particular, a PC cannot consist of only the empty simplex. In fact, the following stronger cardinality conditions must hold:

Proposition 3.3 *Let Σ be a simplicial complex with a complete matching, let n_i be the number of simplices of Σ of length i, and let $n_{i,down}$ be the number of simplices of Σ of length i that are matched with a simplex of length $i-1$. Then $n_{l+1,down} = \sum_{i=0}^{l}(-1)^{l-i}n_i$, for every $l \geq 0$.*

Proof The proof is by induction on l. Let $n_{i,\mathrm{up}}$ be the number of simplices of Σ of length i that are matched with a simplex of length $i+1$. Then $n_{i,\mathrm{up}} = n_{i+1,\mathrm{down}}$. The case $l = 0$ is trivial, because $n_0 = 1$, and exactly one simplex of length 1 is matched to the empty simplex below it. The case $l = 1$ is also trivial, because $n_{2,\mathrm{down}} = n_{1,\mathrm{up}} = n_1 - n_{1,\mathrm{down}} = n_1 - n_0$. The inductive step

$l-1 \to l+1$ for $l \geq 1$ proceeds as follows:

$$\begin{aligned} n_{l+1,\text{down}} &= n_{l,\text{up}} = n_l - n_{l,\text{down}} \\ &= n_l - n_{l-1,\text{up}} = n_l - (n_{l-1} - n_{l-1,\text{down}}) \\ &= n_l - n_{l-1} + n_{l-1,\text{down}} \\ &= n_l - n_{l-1} + \sum_{i=0}^{l-2} (-1)^{l-2-i} n_i \\ &= \sum_{i=0}^{l} (-1)^{l-i} n_i \blacksquare \end{aligned}$$

In Definition 3.1, the partial order $\preceq$ is only used to compare pairs of simplices of the same length that are covered by the same upper simplex. Indeed, it is only used to distinguish $\gamma(\sigma)$ among the simplices covered by the upper simplex σ. To distinguish $\preceq$ from the partial order of the simplicial complex itself, we will sometimes refer to it by using precedence terminology. For example, we will say σ is *earlier* than τ rather than σ is *less* than τ when $\sigma \prec \tau$.

While the precedence partial order on a PC is not unique, the bijection γ determines a unique "smallest" precedence order in a sense we now make precise. Let R be the relation on $\Sigma \times \Sigma$ consisting of the ordered pairs $R = \{(\sigma, \tau) \mid \sigma = \tau \text{ or } \exists \rho \text{ such that } \gamma(\rho) = \sigma \text{ and } \tau \lessdot \rho\}$. Clearly, the bijection γ defines a PC structure if and only if R is acyclic. If R is acyclic, one can define a partial order $\preceq^*$ by taking the transitive closure of R. If we regard partial orders as binary relations related by containment, then a partial order $\preceq$ is compatible with γ if and only if it contains $\preceq^*$. In particular, this means that one could assume that the precedence order is a total order by using a linear extension. We now give an example to show the need for the precedence order.

Example 3.4 *Consider the following simplicial complex, written using the notation introduced in Section 2.3:*

$$\Sigma = \{\emptyset, 0, 1, 2, 3, 01, 02, 12\}.$$

Partition the simplices of Σ *into these two collections:*

$$\{\emptyset, 0, 1, 2\} \text{ and } \{3, 01, 02, 12\}$$

and then arrange them in pairs as follows:

$$(\emptyset, 3), (0, 02), (1, 12), (2, 02).$$

The pairs above have the property that the first simplex in each pair is covered by the second simplex. So Σ *has a complete matching of its simplices.*

It is easy to check that no partial order on the simplices of Σ can serve as the precedence order of Definition 3.1 for the complete matching in Example 3.4. Geometrically, Σ in this example is disconnected, so it does not satisfy the fixed simplex property. Furthermore, the subcomplex of Σ on the vertices $\{0, 1, 2\}$ is connected but does not satisfy the fixed simplex property. For example, the simplicial map that takes each vertex i to $i + 1 \pmod 2$ has no nonempty fixed simplex. So it isn't even true that all of the connected components of Σ satisfy the fixed simplex property.

It is relatively easy to find many more examples of complete matchings of simplicial complexes as well as proper pseudo cones. One can do this by randomly generating simplicial complexes and then checking to see if they have the desired structure. In Figure 1, we show the proportion of simplicial complexes that are proper pseudo cones among all simplicial complexes and among only the ones that satisfy the conditions in Proposition 3.3, respectively. The simplicial complexes are generated by randomly selecting maximal simplices with length L from a set of vertices of size V. There is a graph for each choice of V and L. The horizontal axis is the number of maximal simplices, and the vertical axis is the proportion of the simplicial complexes that are proper pseudo cones. Note that the vertical scale of the graphs on the left differs from that on the right. The software for computing the proportions in Figure 1 is freely available as an executable jar file at [4]. The source code can be extracted from the jar file.

4 Straightening Laws

We now introduce a technique for rewriting a simplex in a standard form. The technique is called a *straightening law*, and the standard form is called the *straightening formula*. The term "straightening" arises from the systematic use of a precedence order to guarantee that the standardization process converges. During a particular straightening operation, it is possible for the same simplex to occur more than once, so it is necessary to keep track of multiplicities.

The straightening law is based on a special kind of path in the Hasse diagram of Σ that we call a *β-path*. In such a path, the edges alternately ascend and descend, such that the ascending edges join simplices that are related to each other by the pseudo cone structure. We make this precise as follows:

Definition 4.1 *Let Σ be a pseudo cone. A* β-path *is a nonempty sequence $\{\sigma_0, \ldots, \sigma_m\}$ of simplices of Σ such that:*

1. *σ_m is an upper simplex.*

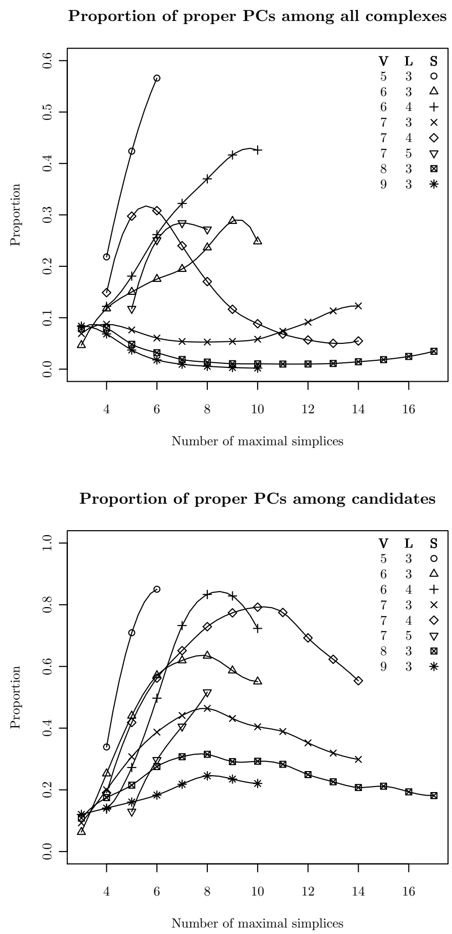


Figure 1: Occurrence of pseudo cones

2. *For every i such that $0 \leq 2i < m$, σ_{2i} is a lower simplex, and $\sigma_{2i+1} = \beta(\sigma_{2i})$;*

3. *For every i such that $0 < 2i \leq m$, $\sigma_{2i} \lessdot \sigma_{2i-1}$, and $\sigma_{2i} \neq \sigma_{2i-2}$.*

The first simplex σ_0 is called the source *of the β-path, while the last simplex σ_m is called the* target *or* destination. *More succinctly, a β-path proceeds* from *its source and* to *its target. The* length *of a β-path is m, i.e., one less than the number of simplices in the sequence.*

$$\begin{tikzcd}
 & \sigma_1 \arrow[dr] & & \sigma_3 \arrow[dr] & & & \cdots & & & \sigma_{m-1} \arrow[dr] & \\
\sigma_0 \arrow[ur, "\beta"] & \prec & \sigma_2 \arrow[ur, "\beta"] & \prec & \sigma_4 & \prec & \cdots & \prec & \sigma_{m-2} \arrow[ur, "\beta"] & \prec & \sigma_m
\end{tikzcd}$$

Figure 2: Even length β-path

$$\begin{tikzcd}
 & \sigma_1 \arrow[dr] & & \sigma_3 \arrow[dr] & & & \cdots & & & \sigma_{m-2} \arrow[dr] & & \sigma_m \\
\sigma_0 \arrow[ur, "\beta"] & \prec & \sigma_2 \arrow[ur, "\beta"] & \prec & \sigma_4 & \prec & \cdots & \prec & \sigma_{m-3} \arrow[ur, "\beta"] & \prec & \sigma_{m-1} \arrow[ur, "\beta"] &
\end{tikzcd}$$

Figure 3: Odd length β-path

Note that by condition (2), the even simplices determine the odd simplices (and vice versa, except for the last simplex when m is even). For a β-path $\{\sigma_0, \ldots, \sigma_m\}$, the simplices with odd subscripts are upper simplices, and they all have the same length. The simplices with even subscripts are lower simplices, except for the last simplex when m is even. The simplices with even subscripts all have the same length. It is easy to see that all the simplices in a β-path are distinct. Furthermore, the simplices having even subscripts form a strictly ascending sequence in the $\preceq$ order. The two kinds of β-path are shown in Figures 2 and 3. A *V-path* in discrete Morse theory (see [14]) is a β-path of even length except that the target is a lower simplex. The "V" in V-path is a discrete vector field. This corresponds to the bijection γ, except that in discrete Morse theory the discrete vector field need not be complete and need not be acyclic. By the discussion prior to Example 3.4, the bijection γ determines a binary relation R. It is easy to check that R is acyclic if and only if there is no closed V-path. Thus a simplicial complex is a PC if and only if it has an acyclic complete matching in the sense of discrete Morse theory.

We now define the straightening formula in terms of β-paths as follows:

Definition 4.2 *Let Σ be a pseudo cone. The* straightening formula *for a simplex σ is the multiset of targets of all β-paths whose source is σ. The straightening formula of σ is written* $\mathrm{str}(\sigma)$.

While the definition of straightening is combinatorial, it can be difficult to prove results using it. The following recursive method for computing the straightening formula is much more useful for proofs based on induction:

Theorem 4.3 *Let Σ be a pseudo cone, and let $\sigma \in \Sigma$ be a simplex.*

1. *If σ is an upper simplex, then* $\mathrm{str}(\sigma) = \{\sigma\}$.
2. *If σ is a lower simplex, then*

$$\mathrm{str}(\sigma) = \{\beta(\sigma)\} + \sum\{\mathrm{str}(\tau) \mid \tau \neq \sigma \textit{ and } \tau \lessdot \beta(\sigma)\}.$$

Proof If σ is an upper simplex, then $\mathrm{str}(\sigma) = \{\sigma\}$ because the only β-path that can have an upper source simplex is a trivial β-path of length 0. Thus part 1 follows.

To show part 2 let σ be a lower simplex. The proof is by induction on the dual of the precedence order. To show the base of the induction, suppose that σ is maximal in the $\preceq$ order. Since σ is a lower simplex, among all simplices covered by $\beta(\sigma)$ it is both a minimum and maximum in the $\preceq$ order. This is only possible for $\emptyset$. The only β-path from $\emptyset$ is $\{\emptyset, \beta(\emptyset)\}$, and it is easy to verify that the result holds in this case.

It remains to show the result in general by induction when σ is a lower simplex. Now all β-paths from σ must pass through $\beta(\sigma)$, and the shortest one ends at $\beta(\sigma)$. The rest of the β-paths either end at an upper simplex covered by $\beta(\sigma)$ or pass through a lower simplex covered by $\beta(\sigma)$ (other than σ). Since the simplices of length $|\sigma|$ in one of the β-paths from σ are strictly increasing in the precedence order, it is not possible for a β-path to "double back." Therefore, the β-paths from σ are partitioned into the following disjoint subsets:

1. A subset consisting of the single path ending at $\beta(\sigma)$;
2. One subset for each of the β-paths ending at one of the upper simplices covered by $\beta(\sigma)$; and
3. One subset for each set of β-paths passing through one of the lower simplices covered by $\beta(\sigma)$.

We now apply the inductive hypothesis on each of these subsets; namely, for each simplex $\tau \neq \sigma$ covered by $\beta(\sigma)$, $\mathrm{str}(\tau)$ is the multiset of targets of paths from τ. Extend each of these paths by prepending σ and $\beta(\sigma)$. This gives the set of all β-paths from σ, and the multiset of targets of those paths is given by the expression in part 2. ■

The most important property of the straightening law is the following parity result which forms the basis for one of the hit pairings used by the combinatorial proof in Section 8.

Theorem 4.4 *Let Σ be a pseudo cone, for any two simplices $\tau, \rho \in \Sigma$ such that $|\tau| = |\rho|$, there is an even number of β-paths starting at a lower simplex which is covered by or equal to ρ and which ends at τ.*

Proof In the proof, we will use the multiset notation defined in Section 2.1. In particular, for a multiset m, the multiplicity of an element x is written $m(x)$, and the multiset union is written using summation notation to emphasize that the multiplicities are being summed. For example, the multiset of targets of β-paths starting at a lower simplex covered by or equal to ρ is $m = \sum_{\theta \,\lessdot\!\!\!=\, \rho} \mathrm{str}(\theta)$. The multiplicity of τ in this multiset is $m(\tau) = \sum_{\theta \,\lessdot\!\!\!=\, \rho} \mathrm{str}(\theta)(\tau)$. This multiplicity is the number of β-paths starting at a lower simplex covered by or equal to ρ and ending at τ. We propose to show that it is even.

The result is trivially true for lower simplices τ since every β-path ends at an upper simplex. Therefore, we may assume that τ is an upper simplex. We will show this by using induction on ρ using the dual of the $\preceq$ order. To show the base of the induction, suppose that ρ is maximal in the $\preceq$ order. As in the proof of Theorem 4.3, ρ is either an upper simplex or is $\emptyset$. We will prove the former case without induction below. In the latter case, τ is also $\emptyset$ because $|\tau| = |\rho|$. However, $\emptyset$ is a lower simplex, and no β-path ends at a lower simplex, so the result is trivially true in this case. Therefore, the base of the induction follows.

To show the general case, we split the computation into two cases depending on whether ρ is upper or lower.

Case 1: ρ is an upper simplex. In this case, $m = \sum_{\theta \,\lessdot\!\!\!=\, \rho} \mathrm{str}(\theta)$ can be written as:

$$m = \mathrm{str}(\rho) + \sum_{\theta \lessdot \rho} \mathrm{str}(\theta) = \{\rho\} + \sum_{\theta \lessdot \rho} \mathrm{str}(\theta) \tag{1}$$

Let $\sigma = \gamma(\rho)$ so that $\rho = \beta(\sigma)$ and $\sigma \lessdot \rho$. Equation (1) may then be written in the following form:

$$m = \{\beta(\sigma)\} + \mathrm{str}(\sigma) + \sum \{\mathrm{str}(\theta) \mid \theta \neq \sigma \text{ and } \theta \lessdot \beta(\sigma)\} \tag{2}$$

By Theorem 4.3 and the fact that σ is a lower simplex, we can express $\mathrm{str}(\sigma)$ as follows:

$$\mathrm{str}(\sigma) = \{\beta(\sigma)\} + \sum\{\mathrm{str}(\theta) \mid \theta \neq \sigma \text{ and } \theta \lessdot \beta(\sigma)\} \tag{3}$$

Substituting equation (3) into equation (2) then gives the following:

$$m = 2(\{\beta(\sigma)\} + \sum\{\mathrm{str}(\theta) \mid \theta \neq \sigma \text{ and } \theta \lessdot \rho\}) \tag{4}$$

Therefore, the multiplicity of any element in m is even, and this case follows.

Case 2: ρ is a lower simplex. Let $\eta = \beta(\rho)$, and construct the following multiset:

$$b = \sum_{\mu \lessdot \eta} \sum_{\theta \lessdot \mu} \mathrm{str}(\theta) \tag{5}$$

Every θ occurring in this double sum differs from η by omitting exactly 2 vertices. There are exactly 2 simplices μ that are between each θ and η, and each such μ omits one of the 2 vertices that are missing from θ. Consequently, in equation (5) every term $\mathrm{str}(\theta)$ occurs exactly twice. In particular, the multiplicity of τ in b is even.

The first summation in equation (5) is over all simplices that are covered by or equal η. One such simplex is ρ because $\eta = \beta(\rho)$. So this summation can be split into the case of $\mu = \rho$ and the case $\mu \neq \rho$ as follows:

$$\begin{aligned} b &= \sum_{\mu \lessdot \eta \text{ and } \mu = \rho} \sum_{\theta \lessdot \mu} \mathrm{str}(\theta) + \sum_{\mu \lessdot \eta \text{ and } \mu \neq \rho} \sum_{\theta \lessdot \mu} \mathrm{str}(\theta) & (6) \\ &= \sum_{\theta \lessdot \rho} \mathrm{str}(\theta) + \sum_{\mu \lessdot \eta \text{ and } \mu \neq \rho} \sum_{\theta \lessdot \mu} \mathrm{str}(\theta) & (7) \end{aligned}$$

We now add $2\,\mathrm{str}(\rho)$ to both sides of equation (7), expand one of the $\mathrm{str}(\rho)$ using Theorem 4.3, and simplify as follows:

$$\begin{aligned} b + 2\,\mathrm{str}(\rho) &= \mathrm{str}(\rho) + \mathrm{str}(\rho) + \sum_{\theta \lessdot \rho} \mathrm{str}(\theta) + \sum_{\mu \lessdot \eta \text{ and } \mu \neq \rho} \sum_{\theta \lessdot \mu} \mathrm{str}(\theta) & (8) \\ &= \mathrm{str}(\rho) + \mathrm{str}(\eta) + \sum_{\mu \lessdot \eta \text{ and } \mu \neq \rho} \mathrm{str}(\mu) & (9) \\ &\quad + \sum_{\theta \lessdot \rho} \mathrm{str}(\theta) + \sum_{\mu \lessdot \eta \text{ and } \mu \neq \rho} \sum_{\theta \lessdot \mu} \mathrm{str}(\theta) & (10) \\ &= \{\eta\} + \sum_{\theta \leq \rho} \mathrm{str}(\theta) + \sum_{\mu \lessdot \eta \text{ and } \mu \neq \rho} \sum_{\theta \leq \mu} \mathrm{str}(\theta) & (11) \end{aligned}$$

where $\mathrm{str}(\eta) = \{\eta\}$ since η is an upper simplex. By definition, $\eta = \beta(\rho)$ so $|\eta| = |\rho| + 1$. Since $|\tau| = |\rho|$, we must have that $\eta \neq \tau$. Therefore, the multiplicity of τ in the multiset $\{\eta\}$ is 0. Consequently, when we compute the multiplicity at τ on both sides of equation (11) we obtain the following:

$$b(\tau) + 2\,\mathrm{str}(\rho)(\tau) = \sum_{\theta \,\leq\, \rho} \mathrm{str}(\theta)(\tau) + \sum_{\mu \lessdot \eta \text{ and } \mu \neq \rho} \sum_{\theta \,\leq\, \mu} \mathrm{str}(\theta)(\tau) \qquad (12)$$

In the double sum occurring in equation (12), every simplex μ occurs later than ρ in the precedence order. We now apply the inductive hypothesis. Recall that we are using the dual precedence order for the induction. Since μ occurs later than ρ, it follows that $\sum_{\theta \,\leq\, \mu} \mathrm{str}(\theta)(\tau)$ is even for every μ occurring in the double sum. We already showed that $b(\tau)$ (i.e., the multiplicity of τ in b) is even in the discussion following equation (5) above. Therefore, every term in equation (12) is known to be even, except for $\sum_{\theta \,\leq\, \rho} \mathrm{str}(\theta)(\tau)$. So this term must also be even. ■

5 Constructing Pseudo Cones

We now show that pseudo cones can be constructed by extending other pseudo cones. This is used in later sections to show that some important classes of posets are pseudo cones.

Theorem 5.1 *Let Σ be a simplicial complex on vertex set V, let $v \in V$ be a vertex, and let $U = V - \{v\}$. If $\Sigma|U$ and $Lk_\Sigma(v)$ are pseudo cones, then Σ is also a pseudo cone.*

Proof Choose PC structures for the simplicial complexes $\Sigma|U$ and $Lk_\Sigma(v)$. Note that although $Lk_\Sigma(v)$ is contained in $\Sigma|U$, we make no assumption that there is any compatibility between the two PC structures. We will be extending the PC structure on $\Sigma|U$ to all of Σ.

We first extend the upper-lower partition. Let $\sigma \in \Sigma$. If $\sigma \subseteq U$, then we label σ as upper or lower the same way as in the PC structure of $\Sigma|U$. Otherwise, $v \in \sigma$, $\sigma - \{v\} \subseteq U$ and $\sigma - \{v\} \in Lk_\Sigma(v)$. Label σ the same way (as upper or lower) that $\sigma - \{v\}$ is labeled in $Lk_\Sigma(v)$.

Next we extend $\preceq$. Let σ and τ be two simplices of Σ. The relation $\sigma \preceq \tau$ is defined to hold if and only if one of the following is true:

1. $\sigma \subseteq U$, $\tau \subseteq U$ and $\sigma \preceq \tau$ in $\Sigma|U$.
2. $\sigma \not\subseteq U$ and $\tau \subseteq U$.

3. $v \in \sigma \cap \tau$ and $\sigma - \{v\} \preceq \tau - \{v\}$ in $Lk_\Sigma(v)$.

In other words, when both simplices are in $\Sigma|U$ or both are not in $\Sigma|U$, then $\preceq$ is induced by $\preceq$ on $\Sigma|U$ or $Lk_\Sigma(v)$, respectively; and every simplex not in $\Sigma|U$ precedes every simplex in $\Sigma|U$. It is easy to check that $\preceq$ defines a partial order on Σ.

Next we check that every upper simplex covers a first lower simplex. Let $\sigma \in \Sigma$ be an upper simplex. If $\sigma \subseteq U$, then the claim follows immediately from the PC structure on $\Sigma|U$. Otherwise, we have that $v \in \sigma$. If there are no simplices covered by σ that contain v, then $\sigma = \{v\}$, and hence σ trivially covers a first lower simplex. So we may assume that σ covers at least one simplex containing v. Among the simplices covered by σ, the ones that do not contain v are later than the ones that contain v, so we only need to show that there is a first simplex covered by σ among the simplices that contain v. By definition of the partition for Σ, $\sigma - \{v\}$ is an upper simplex in $Lk_\Sigma(v)$. By the PC structure on $Lk_\Sigma(v)$, there is a first simplex $\rho \lessdot (\sigma - \{v\})$, and it is a lower simplex. By definition of the PC structure in Σ, $\rho \cup \{v\}$ is a lower simplex in Σ and precedes all other simplices covered by σ and containing $\{v\}$. This is what we wished to show, so it follows that every upper simplex in Σ covers a first lower simplex.

Finally, we need to check that γ is a bijection from the upper simplices to the lower ones. We first check that γ is surjective. Let τ be a lower simplex of Σ.

Suppose that $\tau \subseteq U$. Then τ is also a lower simplex of $\Sigma|U$. Let $\sigma = \beta(\tau)$ in the PC structure of $\Sigma|U$. By definition of the PC structure on Σ, $\tau = \gamma(\sigma)$ holds in the PC structure of Σ.

Next suppose that $v \in \tau$. Then $\tau - \{v\}$ is a lower simplex in $Lk_\Sigma(v)$. Let $\rho = \beta(\tau - \{v\})$ in the PC structure of $Lk_\Sigma(v)$. It is easy to see that $\tau = \gamma(\rho \cup \{v\})$ for the γ of the PC structure of Σ. Therefore the γ of Σ is surjective.

It remains to show that γ is injective. Suppose that σ_1 and σ_2 are two upper simplices such that $\gamma(\sigma_1) = \gamma(\sigma_2)$. Write τ for $\gamma(\sigma_1)$. First suppose that $\tau \subseteq U$. Then $\sigma_1 \subseteq U$ and $\sigma_2 \subseteq U$. The PC structure on $\Sigma|U$ then immediately ensures that $\sigma_1 = \sigma_2$. The other possibility is that $v \in \tau$, in which case $v \in \sigma_1$ and $v \in \sigma_2$. An easy application of the PC structure of $Lk_\Sigma(v)$ ensures that $\sigma_1 - \{v\} = \sigma_2 - \{v\}$ and hence that $\sigma_1 = \sigma_2$. ■

We now generalize Theorem 5.1 to a set W of vertices.

Theorem 5.2 *Let Σ be a simplicial complex on vertex set V, let $W \subseteq V$ be a (possibly empty) set of vertices, and let $U = V - W$. If for every $\sigma \in \Sigma|W$ (including $\sigma = \emptyset$) $St_\Sigma(\sigma)|U$ is a pseudo cone, then Σ is a pseudo cone.*

Proof If W is empty, then $U = V$ and the theorem only asserts that if $St_\Sigma(\emptyset)$ is a PC then Σ is a PC. Since $St_\Sigma(\emptyset) = \Sigma$ this is trivial. Therefore, we may assume that W is nonempty. If U is empty, then one of the hypotheses of the theorem is that $St_\Sigma(\emptyset)|\emptyset$ is a PC, but this is a simplicial complex consisting of only the empty simplex. So the hypotheses cannot be satisfied, and this case is also trivial. Therefore, we may also assume that U is nonempty.

We will use induction on $n = |W|$. The case $n = 1$ is Theorem 5.1, so we may assume that $n > 1$ and that the result is true for $m < n$. Choose a vertex $v \in W$. We propose to show that Σ is a PC by applying Theorem 5.1. To do this we must show that $\Sigma|(V - \{v\})$ and $Lk_\Sigma(v) = St_\Sigma(v)|(V - \{v\})$ are PCs.

First consider the simplicial complex $\Sigma' = \Sigma|(V - \{v\})$. To show that Σ' is a PC we will use the inductive hypothesis with $W' = W - \{v\}$, and $U' = (V - \{v\}) - W' = V - W = U$. To apply the inductive hypothesis we must check that for every $\sigma \in \Sigma'|W'$, we have that $St_{\Sigma'}(\sigma)|U'$ is a PC. It is easy to check that for every such σ, $St_{\Sigma'}(\sigma)|U' = St_\Sigma(\sigma)|U$. Since every such σ is also in $\Sigma|W$, the hypothesis of the theorem implies that $St_{\Sigma'}(\sigma)|U'$ is a PC. Therefore, the inductive hypothesis applies and Σ' is a PC.

Next consider $\Sigma'' = Lk_\Sigma(v)$. Let V'' be the vertex set of Σ''. To show that Σ'' is a PC we will use the inductive hypothesis with $W'' = V'' \cap W'$ and $U'' = V'' - W'' = V'' \cap U$. Since $|W''| \leq |W'| = n - 1 < n$, the inductive hypothesis will apply if for every $\sigma \in \Sigma''|W''$, we have that $St_{\Sigma''}(\sigma)|U''$ is a PC. It is easy to check that for every such σ, $St_{\Sigma''}(\sigma)|U'' = St_\Sigma(\sigma \cup \{v\})|U$. For every such σ, $\sigma \cup \{v\} \in \Sigma|W$, so the hypothesis of the theorem implies that $St_{\Sigma''}(\sigma)|U''$ is a PC. The inductive hypothesis applies so we conclude that Σ'' is a PC. Therefore, by Theorem 5.1, Σ is a PC, and the result follows. ■

We now give a name to the simplicial complexes that can be constructed using the results of this section.

Definition 5.3 *A finite simplicial complex Σ on V is said to be* link reducible *when either Σ consists of a single vertex or there exists $x \in V$ such that*

1. *$\Sigma|(V - \{x\})$ is link reducible, and*
2. *$Lk_\Sigma(x)$ is link reducible.*

A finite poset P is said to be link reducible *when $\Delta(P)$ is link reducible.*

Link reducible simplicial complexes are implicit in [17, Corollary A.4.3] which shows a homological property for such simplicial complexes. Clearly, Theorem 5.2 applies equally well to link reducibility. More precisely,

Corollary 5.4 *Let Σ be a simplicial complex on vertex set V, let $W \subseteq V$ be a (possibly empty) set of vertices, and let $U = V - W$. If for every $\sigma \in \Sigma|W$ (including $\sigma = \emptyset$) $St_\Sigma(\sigma)|U$ is link reducible, then Σ is link reducible.*

Proof In the proof of Theorem 5.2, replace "PC" with "link reducible," and replace references to Theorem 5.1 with references to Definition 5.3. ■

The main significance of link reducibility is the following:

Theorem 5.5 *A link reducible simplicial complex is a pseudo cone.*

Proof Let Σ be a link reducible simplicial complex with vertex set V. We use induction on $n = |V|$. If $n = 1$ then Σ is trivially PC. So we may assume that $n > 1$ and that the result is true for $m < n$. Since $n > 1$, by Definition 5.3 there exists an $x \in V$ such that $\Sigma|(V - \{x\})$ and $Lk_\Sigma(x)$ are link reducible. By the inductive hypothesis, both of these are PCs. By Theorem 5.1, Σ is a PC. ■

The following example shows that the converse of Theorem 5.5 does not hold:

Example 5.6 *Let Σ on $V = \{0, 1, 2, 3, 4, 5\}$ consist of the following simplices:*

$$\{\emptyset, 0, 1, 2, 3, 4, 5, 23, 24, 14, 04, 03, 05, 01, 02, 13, 35, 34, 12,$$
$$15, 25, 45, 012, 014, 124, 034, 234, 125, 025, 135, 035, 345\}$$

Then Σ is a proper pseudo cone with the precedence order as shown. However, for every vertex x, the number of simplices in $\Sigma|(V - \{x\})$ is 21, and the number of simplices in $Lk_\Sigma(x)$ is 11. Because a PC must have an even number of simplices, none of the subcomplexes and links is a PC, and hence none is link reducible. Therefore, Σ is not link reducible.

6 Dismantlable and Collapsible Ordered Sets

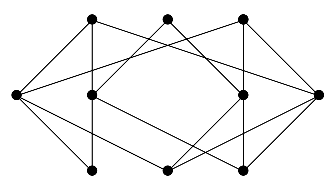
As an application of the results of Section 5, we examine combinatorial notions of dismantlability and collapsibility. These notions strengthen the concept of link reducibility by adding a retraction condition. A *retraction* is a map $f : T \to S$ where S is a subset of T and $f|S$ is the identity on S. There are various notions of retraction depending on what kind of map f is required to be. We begin with a concept that applies to any finite simplicial complex. Later, we will introduce concepts that apply only to finite posets.

Definition 6.1 *A finite simplicial complex Σ on V is said to be* link collapsible *when either Σ consists of a single vertex or there exists $x \in V$ such that*

1. *There exists a simplicial retraction from V to $V - \{x\}$,*
2. *$\Sigma|(V - \{x\})$ is link collapsible, and*
3. *$Lk_\Sigma(x)$ is link collapsible.*

A finite poset P is said to be link collapsible *when $\Delta(P)$ is link collapsible. Note that for $x \in P$, $Lk_{\Delta(P)}(x)$ is the order complex of $V(x) \cup J(x) - \{x\}$.*

Example 6.2 *It is easy to find and/or verify examples of posets and simplicial complexes that are link reducible but not link collapsible by using the software at [4]. The following is one such example:*



Proposition 6.3 *A finite cone is link collapsible.*

Proof Let Σ be a cone with peak x, and vertex set V. We use induction on the number n of vertices of Σ. If Σ consists only of x, then it is link collapsible by definition. So we may assume that $n > 1$ and that the result is true for cones with $m < n$ vertices. Choose any $y \in V$ that is not the same as x. Such an element exists because $n > 1$. Now both $\Sigma' = \Sigma|(V - \{y\})$ and $Lk_\Sigma(y)$ contain x and so are cones. Both of these cones have fewer vertices than Σ (since they do not contain y), so they are link collapsible by the inductive hypothesis. It remains to find a retraction r from Σ to Σ'. By definition of a retraction, $r(v)$ must be v for $v \neq y$. Define $r(y)$ to be x. Let $\sigma \in \Sigma$. If $y \notin \sigma$, then σ is in Σ' and $r(\sigma) = \sigma$. If $y \in \sigma$, then $r(\sigma) = (\sigma - \{y\}) \cup \{x\}$. Now Σ is a cone with peak x so $\sigma \cup \{x\} \in \Sigma$. Hence the subsimplex $(\sigma - \{y\}) \cup \{x\}$ is also in Σ. Since y is not in $(\sigma - \{y\}) \cup \{x\}$, it is in Σ'. Therefore, r is a simplicial map from Σ to Σ', and the result follows. ■

For the rest of the section, we restrict attention to posets. In the definition of link collapsibility, if the simplicial complex is an order complex, then the simplicial retraction need not be order preserving. By requiring that the retraction be order-preserving, we obtain the notion of connected collapsibility defined in [17, Definition 4.3.12]. More precisely,

Definition 6.4 *A finite poset P is said to be* connectedly collapsible *when either $|P| = 1$ or there exists $x \in P$ such that*

1. *there exists an order-preserving retraction from P to $P - \{x\}$,*
2. *$P - \{x\}$ is connectedly collapsible, and*
3. *$V(x) \cup J(x) - \{x\}$ is connectedly collapsible.*

The notion of connected collapsibility was the motivation for introducing the notion of link collapsibility. The notions differ in two ways. First, link collapsibility applies to all finite simplicial complexes, while connected collapsibility applies only to finite posets. Second, a simplicial retraction from P to $P - \{x\}$ need not be order-preserving. However, an order-preserving map induces a simplicial map on the order complexes, so if P is connectedly collapsible then P is link collapsible. Surprisingly, the converse also holds.

Proposition 6.5 *A finite poset is connectedly collapsible if and only if it is link collapsible.*

Proof As noted above, connectedly collapsible trivially implies link collapsible, so we only need to show the converse. Let P be a link collapsible poset. We will prove the result by induction on $|P|$. If $|P| = 1$, then P is trivially connectedly collapsible so the base of the induction holds.

For the induction step, suppose that $|P| > 1$ and that the converse holds for posets with fewer elements than P. Then there is an $x \in P$ such that there is a simplicial retraction $r\colon P \to P - \{x\}$ for which $P - \{x\}$ and $V(x) \cup J(x) - \{x\}$ are link reducible. By the inductive hypothesis, both of these are connectedly collapsible. Let $y = r(x)$. Since r is a simplicial retraction, it is order-preserving for elements in $P - \{x\}$, and if x is comparable with $z \in P - \{x\}$ then y is comparable with z.

We first consider the case in which x and y are not comparable. If $x < z$, then it cannot be the case that $y \geq z$ so one must have $y < z$. Similarly, if $x > z$ in P, then $y > z$. Therefore, r is an order-preserving retraction, and P is connectedly collapsible.

Next suppose that $x \gtrdot y$. If $x < z$, then it is immediate that $y \leq z$. If $x > z$ in P, then one cannot have $y < z$ so one must have $y \geq z$. Therefore, r is an order-preserving retraction, and P is connectedly collapsible. Similarly, the result follows if $y \gtrdot x$.

Therefore, we may assume that x and y are comparable and that neither covers the other. Without loss of generality we may assume that $x < y$. By our assumptions, there exists z such that $x < z \lessdot y$. Now suppose that $w \gtrdot z$. Then w is comparable with x so w is also comparable with y, but w and y both cover z so it follows that $w = y$ and hence y is the only cover of z. Clearly, mapping z to y defines an order-preserving retraction $P \to P - \{z\}$. In [16, Theorem 2] it is shown that a retraction of a connectedly collapsible clique complex is also a connectedly collapsible clique complex. As noted in Section 2.4, an order complex is a special case of a clique complex. Therefore, $P - \{z\}$ is link collapsible. By the inductive hypothesis, it is also connectedly collapsible. Now consider the link of z; namely, $V(z) \cup J(z) - \{z\}$. Since y is the only cover of z, $V(z) \cup J(z) - \{z\}$ is a cone with peak y. Therefore, by Proposition 6.3, $V(z) \cup J(z) - \{z\}$ is link collapsible. By the inductive hypothesis, it is also connectedly collapsible. It follows that P is connectedly collapsible. ■

The last notion we consider is dismantlability. This was one of the kinds of poset that was shown in [5] to have the fixed point property and for which the problem of finding a combinatorial proof was posed. A finite-length poset P is said to be *dismantlable* if the identity function on P is in the same connected component of P^P as a constant function in P^P. See [17, Section 4.3] for a more general notion of dismantlability. By [5, Theorem 4.1], a finite poset is dismantlable if and only if it is reducible to a one-element poset by removing a sequence of irreducibles, where an *irreducible* is an element of P that is covered by or covers exactly one element. As usual, the first case we consider is that of a cone.

Proposition 6.6 *A finite poset cone is dismantlable.*

Proof Let P be a poset cone with peak x. We use induction on the number n of elements of P. If P consists only of x, then it is dismantlable by definition. So we may assume that $n > 1$ and that the result is true for poset cones with $m < n$ elements. Since x is a peak, and $n > 1$, there exists an element y such that either $x \lessdot y$ or $y \lessdot x$. Clearly y is irreducible. Now $P - \{y\}$ contains x, so it is a cone. Since it has fewer elements than P, it is dismantlable. The result then follows by induction. ■

We now show that dismantlability is stronger than connected collapsibility and consequently a finite dismantlable poset is a PC.

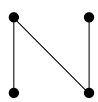
Proposition 6.7 *A finite dismantlable ordered set is connectedly collapsible.*

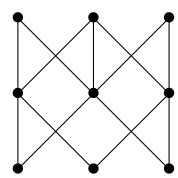
Proof Let P be a finite dismantlable ordered set. We use induction on $|P|$. The result is trivial when $|P| = 1$, so we may assume that $|P| > 1$. As noted above, there exists an irreducible $x \in P$ such that $P - \{x\}$ is dismantlable. By the inductive hypothesis, $P - \{x\}$ is connectedly collapsible. Moreover, it is easy to see that mapping x to the unique element that it covers or is covered by is an order-preserving retraction from P to $P - \{x\}$.

It remains to check that $Q = J(x) \cup V(x) - \{x\}$ is connectedly collapsible. If $y \lessdot x$ then $J(x) = J(y) \cup \{x\}$, so that Q is a cone with peak y. Dually, if $x \lessdot y$, then the same argument applies using $V(x)$ instead of $J(x)$. By Proposition 6.3, Q is link collapsible; and Proposition 6.5 implies that Q is connectedly collapsible. ■

One might think that Proposition 6.3 is a consequence of Proposition 6.6 and Proposition 6.7. However, Proposition 6.3 applies to general finite simplicial complexes, while the other two propositions only apply to finite posets.

Example 6.8 *Neither Proposition 6.6 nor Proposition 6.7 have converses. The first example below is dismantlable but not a poset cone; the second example is connectedly collapsible but not dismantlable.*





The relationships between the various notions in this section are summarized in the following:

Corollary 6.9 *For a finite simplicial complex, cone ⇒ link collapsible ⇒ link reducible ⇒ pseudo cone ⇒ has a complete matching.*
For a finite poset, cone ⇒ dismantlable ⇒ connectedly collapsible ⇔ link collapsible ⇒ link reducible ⇒ pseudo cone ⇒ order complex has a complete matching.
Except as indicated, none of the converses hold.

Proof For a simplicial complex, the first implication is Proposition 6.3; the second follows trivially by definition; the third is Theorem 5.5; and the fourth follows by definition. That the first implication does not have a converse is shown by either part of Example 6.8; the second is shown in Example 6.2; the third is shown in Example 5.6; and the fourth is shown in Example 3.4.

For a poset, the first implication is Proposition 6.6; the second is Proposition 6.7; the third is Proposition 6.5; the fourth follows trivially by definition;

the fifth is a consequence of Theorem 5.5; and the sixth follows by definition. That the first two implications do not have converses is shown in Example 6.8; the fourth is shown in Example 6.2; the fifth is shown in Example 5.6; and the sixth is shown in Example 3.4. ■

7 Truncated Lattices

Let L be a finite lattice. It is easy to check that such a lattice has a maximum element $\hat{1}$ and a minimum element $\hat{0}$. One of the main results of [2, Corollary 6.3] is that the truncation of a finite, noncomplemented lattice is $\mathbb{Q}$-acyclic. This result was motivated by Crapo's computation of the Möbius function of such a lattice in [12]. The techniques of this section are based on those introduced by Crapo in his paper and also by Rota in [15].

It follows from the acyclicity result that $\breve{L}$ has the fixed point property. In [6] a more general class of posets obtained from finite lattices is shown to be $\mathbb{Q}$-acyclic. We now strengthen this result by showing that these posets are link reducible and hence pseudo cones.

Theorem 7.1 *Let L be a finite lattice. Let $x \in \breve{L}$ be any element of the truncated lattice, and let B be any subset of $\breve{L}$ that contains all the complements of x and is contained in the set of lower semi-complements of x. Then $\breve{L} - B$ is link reducible. In particular, if x is noncomplemented, then B can be chosen to be empty so that $\breve{L}$ is itself link reducible.*

Proof Let $C = \{y \in \breve{L} \mid x \wedge y = \hat{0} \text{ and } x \vee y = \hat{1}\}$ be the set of complements of x, and let $S = \{y \in \breve{L} \mid x \wedge y = \hat{0}\}$ be the set of lower semi-complements of x. The set B is a set between these two, i.e., $C \subseteq B \subseteq S$. Note that any or all of these three sets could be empty, and B could be equal to C, S or both.

Let $P = \breve{L} - B$. We propose to apply Corollary 5.4 to $\Sigma = \Delta(P)$, with $W = S - B$ and $U = \breve{L} - S$ so that $P = W \cup U$. Note that U is a filter while W is an ideal. We must show that for every chain $\sigma \in \Delta(W)$, including the empty chain, $St_{\Delta(P)}(\sigma)|U$ is link reducible.

We first consider the case of an empty chain. Now $St_{\Delta(P)}(\emptyset)|U = \Delta(U)$, so we need to show that the poset U is link reducible. Define a function $f\colon U \to U$ by $f(z) = x \wedge z$. To show that f is well-defined, let $z \in U$. Since $U = \breve{L} - S$, $x \wedge z > \hat{0}$. Since $x \wedge z \leq x < \hat{1}$, it follows that $f(z) \in \breve{L}$. Now $x \geq x \wedge z = f(z)$ implies that $f(z) \wedge x = f(z)$. Since $f(z) > \hat{0}$, we have that $f(z) \notin S$. Therefore, $f(z) \in U$ and hence f is well-defined. Clearly, f is order-preserving. Let f_0 be the identity function on U, and let f_2 be the constant function with value

x. It is easy to check that $f_0 \geq f \leq f_2$. Therefore U is dismantlable. By Proposition 6.7, U is link reducible. If W is empty, then $P = U$ and the result follows. So for the rest of the proof we may assume that W is nonempty.

Let σ be a nonempty simplex of $\Delta(W)$. Then σ is a chain $\{w_1 < \cdots < w_n\}$ of elements of W. Now $St_{\Delta(P)}(\sigma)|U$ consists of those chains τ of U such that $\sigma \cup \tau$ is a chain of P. Since U is a filter, τ is such a chain if and only if $\tau \subseteq V(w_n) \cap U$. Therefore $St_{\Delta(P)}(\sigma)|U = \Delta(V(w_n) \cap U)$. So we need to show that for every $w \in W$, the poset $V(w) \cap U$ is link reducible.

Recall that $C \subseteq B \subseteq S$ and that $W = S - B$. Let $w \in W$. Then w is a lower semi-complement but not a complement of x. So $w \vee x \neq \hat{1}$. Since $w \vee x \geq x$, we have that $w \vee x$ not a semi-complement of x. Therefore $w \vee x \in V(w) \cap U$. Define a function $g\colon V(w) \cap U \to V(w) \cap U$ by $g(z) = z \wedge (w \vee x)$. To see that g is well-defined, let $z \in V(w) \cap U$. Now $z \geq w$ and $w \vee x \geq w$ so $g(z) = z \wedge (w \vee x) \geq w$. Hence $g(z) \in V(w)$. Next compute $g(z) \wedge x = (z \wedge (w \vee x)) \wedge x = z \wedge ((w \vee x) \wedge x) = z \wedge x \neq \hat{0}$ because $z \in U$. Therefore, $g(z) \in U$ and g is well-defined. Clearly, g is order-preserving. Now for any $z \in V(w) \cap U$, $z \geq z \wedge (w \vee x) \leq w \vee x$, so g connects the identity function with the constant function having value $w \vee x$. Therefore $V(w) \cap U$ is dismantlable. By Proposition 6.7, $V(w) \cap U$ is link reducible for every $w \in W$. The result then follows from Corollary 5.4. ■

Note that the approach in [10] can also be used to show that truncated noncomplemented lattices are pseudo cones.

Let L be a finite lattice and $x \in L$. We say that y is a *lower strong* complement of x if x and y are complements and y is the join of some set of minimal elements of $\breve{L}$. One dually defines an *upper strong* complement. A finite lattice is *strongly complemented* if every element $x \in \breve{L}$ has both an upper strong complement and a lower strong complement. (See [8] for more general definitions.)

Theorem 7.2 *Let L be a finite lattice that is not strongly complemented. Then $\breve{L}$ is link reducible.*

Proof Let $x \in \breve{L}$ be an element that does not have both an upper strong and a lower strong complement. Without loss of generality, we may assume that x does not have a lower strong complement.

Let C be the set of ordinary complements of x. By Theorem 7.1, we know that $\breve{L} - C$ is link reducible. Let C_1 be the set of minimal elements of C; let C_2 be the minimal elements of $C - C_1$; and so on. Since L is finite, C_n is empty for some n, and $C = \bigcup_{i=1}^{\infty} C_i$ is a finite union. We will show by induction that $L_m = \breve{L} - \bigcup_{i=m}^{\infty} C_i$ is link reducible for every $m \geq 1$, and we have already shown the case $m = 1$.

Let $y \in C_m$ for some $m \geq 1$. Since y is a complement of x, and since x does not have a lower strong complement, y cannot be a join of minimal elements of $\check{L}$. Let A be the set of minimal elements of $J(y)$, and let z be the join of A. Then $z < y$. Since C_m is the set of minimal elements of $\bigcup_{i=m}^{\infty} C_i$, it follows that $z \in L_m$. Let Q be the set of all elements of L_m that are comparable with y. Define an order-preserving self-map $f\colon Q \to Q$ by $f(q) = q \wedge z$. To see that f is well defined, let $q \in Q$. Clearly $f(q) \leq z < y$, so we only need to show that $f(q) > \hat{0}$. If $q > y$, then $q > z$ and $f(q) = z > \hat{0}$. If $q < y$, then $q \geq a$ for some $a \in A$ and $f(q) = q \wedge z \geq a \wedge z = a > \hat{0}$. Therefore f is well defined. It is clearly order-preserving. Now for any $q \in Q$, $q \geq q \wedge z \leq z$, so f connects the identity function with a constant function. Therefore Q is dismantlable.

We now apply Corollary 5.4 to $\Delta(L_{m+1})$ with $W = C_m$ and $U = L_m$. We need to show that $St_{\Delta(L_{m+1})}(\sigma)|U$ is link reducible for every chain σ of $W = C_m$. Since no two elements of C_m are comparable, σ is either empty or a single element. If σ is empty, then the star is the order complex of $U = L_m$ which is link reducible by the inductive hypothesis. If $\sigma = \{y\}$, then the star is the order complex of Q which is dismantlable and hence link reducible. Therefore, L_{m+1} is link reducible, and by induction, $\check{L}$ is link reducible. ■

8 The Combinatorial Proof of the Fixed Simplex Property

The main result of this section is a direct combinatorial proof that finite pseudo cones have the fixed simplex property. We first define the notion that forms the basis of the proof.

Definition 8.1 *Let Σ be a pseudo cone, and let f be a simplicial self-map. A* hit *is an ordered pair (σ, τ) such that:*

1. *$\sigma \leq \tau$ in Σ;*
2. *$|f(\sigma)| = |\sigma|$; and*
3. *$\tau \in \mathrm{str}(f(\sigma))$.*

Note that by the last condition above, τ must be an upper simplex. The multiplicity of a hit is the multiplicity of τ in $\mathrm{str}(f(\sigma))$.

A hit is a coincidence that is weaker than a fixed simplex.

Proposition 8.2 *Let Σ be a pseudo cone, and let f be any simplicial self-map. If σ is a fixed simplex of f then exactly one of the following conditions holds:*

1. *σ is an upper simplex and (σ, σ) is a hit, or*
2. *σ is a lower simplex and $(\sigma, \beta(\sigma))$ is a hit.*

Conversely, if (σ, τ) is a hit and σ is a fixed simplex of f, then τ is an upper simplex which equals either σ or $\beta(\sigma)$.

Proof Let σ be a fixed simplex of f. If σ is an upper simplex, then $\mathrm{str}(f(\sigma)) = \mathrm{str}(\sigma) = \{\sigma\}$. By Definition 8.1, (σ, σ) is a hit. If σ is a lower simplex, then $\mathrm{str}(f(\sigma)) = \mathrm{str}(\sigma) = \{\beta(\sigma)\} + \cdots$ by Theorem 4.3. Therefore, $\beta(\sigma) \in \mathrm{str}(f(\sigma))$, so that $(\sigma, \beta(\sigma))$ is a hit.

Conversely, suppose that (σ, τ) is a hit and that $f(\sigma) = \sigma$. If σ is an upper simplex, then $\mathrm{str}(f(\sigma)) = \mathrm{str}(\sigma) = \{\sigma\}$ as above, so the only possibility for τ is σ. If σ is a lower simplex, then $\mathrm{str}(f(\sigma)) = \mathrm{str}(\sigma)$ may have many simplices whose length is $|\sigma| + 1$. Because (σ, τ) is a hit, we know that $\tau \in \mathrm{str}(\sigma)$. Let $\eta = \gamma(\tau)$. Then η precedes every simplex covered by τ in the $\preceq$ order. In particular, η precedes or equals σ. However, σ precedes or equals every lower simplex of length $|\sigma|$ that occurs in $\mathrm{str}(\sigma)$. Therefore, $\eta = \sigma$ and $\tau = \beta(\sigma)$, and the result follows. ■

Proposition 8.3 *Let Σ be a pseudo cone, and let f be a simplicial self-map. A pair (σ, τ) of simplices is a hit if and only if $\sigma \leq \tau$, $|f(\sigma)| = |\sigma|$, and there is a β-path from $f(\sigma)$ to τ. Furthermore, the multiplicity of a hit (σ, τ) is the number of β-paths from $f(\sigma)$ to τ.*

Proof The result follows immediately from Definition 4.1. ■

The combinatorial proof of the fixed simplex property is based on partitions of hits into pairs. The existence of such pairings depends on knowing that particular sets of hits have even cardinality. The basis for the combinatorial proof is the following parity result:

Theorem 8.4 *Let Σ be a pseudo cone, and f a simplicial self-map of Σ.*

1. *For every simplex $\tau \in \Sigma$, the number of hits (counting multiplicities) that have τ as their second coordinate is always even.*

2. *For every simplex $\sigma \in \Sigma$, the number of hits (counting multiplicities) that have σ as their first coordinate is odd if and only if σ is a fixed simplex of f.*

Proof We first show part 1. Let $\tau \in \Sigma$ and let $l = |\tau|$. There are exactly $l+1$ simplices that are covered by or equal to τ. To show the parity condition, we consider the following cases:

1. $|f(\tau)| < l-1$. Then $|f(\sigma)| < |\sigma|$, for any $\sigma \lessdot \tau$, and there are no hits having τ as the second coordinate.
2. $|f(\tau)| = l-1$. Then $|f(\sigma)| = |\sigma|$ for exactly two simplices σ_1 and σ_2 that are covered by τ. Moreover, $f(\sigma_1) = f(\sigma_2) = f(\tau)$. So the multiplicity of the hit (σ_1, τ) is the same as the multiplicity of the hit (σ_2, τ). It follows that the total number is even.
3. $|f(\tau)| = l$. Then f is bijective on the simplices contained in τ. In particular, the set $\{\theta \mid \theta \lessdot f(\tau)\}$ is the same as the set $\{f(\sigma) \mid \sigma \lessdot \tau\}$. Applying Theorem 4.4 with $\rho = f(\tau)$ gives that $\sum_{\theta \lessdot f(\tau)} \mathrm{str}(\theta)(\tau)$ is even and hence $\sum_{\sigma \lessdot \tau} \mathrm{str}(f(\sigma))(\tau)$ is even. Since every $\sigma \lessdot \tau$ has the property that $|f(\sigma)| = |\sigma|$, the sum of the multiplicities of the hits of the form (σ, τ) is even.
4. $|f(\tau)| > l$. This is impossible because f is a simplicial map.

These are all the cases, so part 1 holds.

We now show part 2. Let $\sigma \in \Sigma$. For every $\rho \in \Sigma$ such that $|\rho| = |\sigma|$, define $Q(\sigma, \rho) = \{\tau \mid \sigma \subseteq \tau \text{ and } \tau \in \mathrm{str}(\rho)\}$, where the multiplicity of τ in $Q(\sigma, \rho)$ is the multiplicity of τ in $\mathrm{str}(\rho)$. We will show that $Q(\sigma, \rho)$ has odd cardinality if and only if $\sigma = \rho$.

We first consider the case in which $\sigma = \rho$, i.e., the parity of $Q(\sigma, \sigma)$. If σ is an upper simplex, then $\mathrm{str}(\rho) = \mathrm{str}(\sigma) = \{\sigma\}$. Therefore, $Q(\sigma, \sigma) = \{\sigma\}$ in this case. Now let σ be a lower simplex and let $\tau \in Q(\sigma, \sigma)$. Then τ is an upper simplex and so cannot coincide with σ. Thus $\sigma \lessdot \tau$. However, σ is the first simplex in the $\preceq$ order among all of the lower simplices covered by the upper simplices in $\mathrm{str}(\rho) = \mathrm{str}(\sigma)$. Therefore $\gamma(\tau) = \sigma$ and $\tau = \beta(\sigma)$, and hence $Q(\sigma, \sigma) = \{\beta(\sigma)\}$. We conclude that $|Q(\sigma, \sigma)|$ is always odd, and in fact, always 1.

It remains to determine the parity of $Q(\sigma, \rho)$ in general. We do this by induction on the cardinality of the multiset $\mathrm{str}(\rho)$. Since $|\mathrm{str}(\rho)| \geq 1$, the base of the induction is the case $|\mathrm{str}(\rho)| = 1$, which occurs in just two ways. The first is when ρ is an upper simplex, and the second is when $\rho = \emptyset$. In the former case, the only choice for τ is ρ, so that $\sigma \subseteq \rho$. Since $|\rho| = |\sigma|$, we must have that $\sigma = \rho$, and we have already computed $Q(\sigma, \sigma)$. In the latter case, since $|\sigma| = |\rho|$, $\sigma = \emptyset$, and we have already computed $Q(\emptyset, \emptyset)$. Accordingly, we may assume that $|\operatorname{str}(\rho)| > 1$, that $\sigma \neq \rho$, and that ρ is lower.

By Theorem 4.3, $\operatorname{str}(\rho)$ is the multiset union $\{\beta(\rho)\} + (\sum_{i=2}^{l} \operatorname{str}(\zeta_i))$, where $l = |\beta(\rho)|$ and $\zeta_1 = \rho, \zeta_2, \ldots, \zeta_l$ are the simplices covered by $\beta(\rho)$. Therefore,

$$Q(\sigma, \rho) = \begin{cases} \{\beta(\rho)\} + (\sum_{i=2}^{l} Q(\sigma, \zeta_i)), & \text{if } \sigma \subseteq \beta(\rho), \\ \sum_{i=2}^{l} Q(\sigma, \zeta_i), & \text{if } \sigma \nsubseteq \beta(\rho). \end{cases}$$

Since $|\operatorname{str}(\zeta_i)| < |\operatorname{str}(\rho)|$ for any $i \geq 2$, the inductive hypothesis implies that $Q(\sigma, \zeta_i)$ has odd cardinality if and only if $\sigma = \zeta_i$. To compute the parity of $Q(\sigma, \rho)$, there are two cases to consider:

1. $\sigma \subseteq \beta(\rho)$. Since we have assumed that $\sigma \neq \rho$, we have that $\sigma = \zeta_j$ for some $j > 1$. In this case, $\sigma \neq \zeta_i$ for $i \neq j$. So $Q(\sigma, \zeta_j)$ has odd cardinality, and $Q(\sigma, \zeta_i)$ has even cardinality for $i \neq j$ by induction. Therefore, $Q(\sigma, \rho)$ has even cardinality as desired.
2. $\sigma \nsubseteq \beta(\rho)$. This implies that $\sigma \neq \zeta_i$ for any $i > 1$, so every $Q(\sigma, \zeta_i)$ for $i > 1$ has even cardinality by induction, and hence $Q(\sigma, \rho)$ does also.

Therefore, $Q(\sigma, \rho)$ has odd cardinality if and only if $\sigma = \rho$. We now use this result to characterize the hits having first coordinate σ. If $|f(\sigma)| < |\sigma|$, then there are no hits, so the result follows in this case. If $|f(\sigma)| = |\sigma|$, then the multiset of hits is $\{(\sigma, \tau) \mid \tau \in Q(\sigma, f(\sigma))\}$. As we have just shown, this has odd cardinality if and only if $\sigma = f(\sigma)$, and so part 2 holds. ■

It is easy to see that the parities in Theorem 8.4 are unaffected by the expedient of removing all hits that have even multiplicity and replacing each hit with odd multiplicity by a single hit. We will abuse notation and refer to such hits as *odd hits*. The odd hits form a graph whose vertices are the odd hits and the edges are pairs of hits that share a first or second component. We call this the *hit graph*.

Theorem 8.5 *Let Σ be a finite pseudo cone, and f a simplicial self-map of Σ. Let G be the hit graph for Σ and f. Then there is a nonempty collection of hit-disjoint paths in G whose endpoints are hits of the form (σ, τ) where σ is a fixed simplex. In particular, Σ has at least one nonempty fixed simplex.*

Proof We begin by choosing subsets F_1 and F_2 of the edges of G. For a simplex $\tau \in \Sigma$, the set $\{(\sigma, \tau) \text{ is an odd hit}\}$ is even. Choose a partition of this set into 2-element subsets. Each 2-element subset is an edge of G. The subset F_1 consists of all such edges for every simplex $\tau \in \Sigma$. Similarly, for a simplex $\sigma \in \Sigma$ that is not fixed by f, the set $\{(\sigma, \tau)$ is an odd hit$\}$ is even. Choose partitions as before to form the the set F_2 of edges. Finally, for a simplex σ that is fixed by f, the multiset $\{(\sigma, \tau)$ is a hit$\}$ has exactly one element by Proposition 8.2, so the notions of hit and odd hit coincide in this case. Such an odd hit is not on any edge of F_2, and such odd hits correspond bijectively with the fixed simplices of f. Because of this bijection, we will abuse notation and refer to these hits as "fixed simplices."

Now it is easy to check that $F_1 \cap F_2 = \emptyset$. Let $F = F_1 \cup F_2$. By construction, every odd hit occurs on exactly one edge of F_1, and every odd hit that is not a fixed simplex occurs on exactly one edge of F_2. Thus a fixed simplex is incident on exactly one edge of F while all other odd hits are incident on exactly two edges of F. It is easy to see that the connected components of F are of two kinds. If a connected component has no fixed simplices, then it forms a cycle. If a connected component has at least one fixed simplex, then it must necessarily have exactly two of them, and they are the endpoints of a path in the graph. These are the paths required by the theorem. It might be worth mentioning that this last part of the proof could be regarded as going back to Euler, who showed that the Seven Bridges of Königberg problem could not be solved by using a similar argument[11]. In any case, there is always exactly one "trivial" fixed simplex; namely, the empty simplex. Therefore, there is at least one path. The result then follows. ■

For either kind of connected component of the graph F in the proof above, the edges alternate between those in F_1 and F_2. So the cycles always have an even number of edges, and the paths have an odd number of edges. If the self-map f is the identity map, then every simplex is fixed, so every connected component of F consists of a single edge. These edges correspond bijectively with the complete matching of the pseudo cone structure.

Corollary 8.6 *Every finite PC ordered set has the fixed point property.*

Proof This follows immediately from Theorem 8.5 and Proposition 2.2. ■

Theorem 8.5 and Corollary 8.6 give the promised combinatorial proof that finite PC posets have the fixed point property. In particular, we obtain combinatorial proofs that finite dismantlable ordered sets and truncated noncomplemented lattices have the fixed point property. This resolves the open problem stated in [5].

9 Fixed Point Sets

In general, one cannot say very much about the structure of of the fixed point set of an order-preserving self-map. For example, the fixed point set of an acyclic poset need not be acyclic. This was shown in [6, Example 2.2]. About all that one can say in general about a fixed point set of an acyclic poset is that its Möbius function (reduced Euler characteristic) is zero.

However, when one has an additional structure on a poset, then it may be possible to show that the fixed point set also has this structure. This is true for poset cones. To see why, let P be a poset cone with peak p, and let f be an order-preserving self-map of P. Since p is a peak, $f(p)$ is comparable with p, say $f(p) \geq p$. By the Abian-Brown Theorem[1], there is a smallest fixed point above p and all fixed points will be comparable to it. Similarly for the case $f(p) \leq p$. So the fixed point set is a poset cone.

10 Acknowledgments

I wish to acknowledge Gian-Carlo Rota for his question to me about the "combinatorial meaning" of the Leray spectral sequence which started the research in which I found the acyclicity result in [2] from which it was found that finite truncated noncomplemented lattices have the fixed point property. I also wish to acknowledge Anders Björner who collaborated with me on the development of the theory. It was in our paper on fixed points in partially ordered sets [5] that we posed the problem of finding combinatorial proofs of our fixed point theorems. I found the main ideas for the combinatorial proofs in 1996. At the Festscrift in honor of Gian-Carlo Rota in 1996, I presented the concept of a pseudo cone and sketched the combinatorial proofs.

Special thanks to the referees of my original submission who found quite a few misprints, and who made a number of valuable suggestions. It was the referees who pointed out the connections with the work of Forman on discrete Morse theory[13], the work of B. Schröder[17], and the work of Constantin and Fournier[10]. In addition, one of the referees proved both Theorem 5.5 and Proposition 6.5 and also suggested that dismantlable posets are connectedly collapsible, not just pseudo cones. And finally, I wish to thank Jonathan Farley for encouraging me to publish my results.

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