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# Rings with Lexicographic Straightening Law*

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## 1. INTRODUCTION

The concept of a "straightening law" as a means of analyzing the structure of particular commutative algebras has appeared independently in the work of a number of authors. The purpose of this paper is to introduce a systematic theory of algebras endowed with a structure of a lexicographic straightening law based on a partially ordered set (or "lexicographic ring" for short). Much of the work in the literature dealing with such rings fits quite naturally in the context of our theory, and we sketch some of this work. Moreover, some of our later results, in particular the Betti number bound (in Section 6), may actually be new results even for the special cases of the known examples of lexicographic rings.

The concept of a lexicographic ring as an interesting area of study was suggested to this author by DeConcini, who conjectured that a lexicographic ring is Cohen–Macaulay if the underlying partially ordered set is so. We prove this result in two different ways. It has recently come to our attention that this conjecture has also been proved by DeConcini *et al.* [13], using deformation theory methods.

The results of this paper are arranged in two parts. The first part, consisting of Sections 2 through 4, is more elementary and uses the technique of "combinatorial decompositions" as developed by Garsia and this author. The second part, Sections 6 and 7, requires some knowledge of homological algebra methods in ring theory. The intermediate Section 5 discusses some of the known examples of lexicographic rings. In this section we also sketch some of the ways that our theory may be extended to more general contexts.

The main result of the first part is the Transfer Theorem 4.3, which enables one to transfer combinatorial decompositions from the underlying partially ordered set (or more precisely from the corresponding Stanley–Reisner ring) to the lexicographic ring. As a result those concepts that are expressible in terms of combinatorial decompositions are also

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transferred, as, for example, the Cohen–Macaulay property and regular sequences may be transferred to the lexicographic ring. As a result the theory of Cohen–Macaulay ordered sets, introduced by this author and developed by Stanley, Reisner, Hochster, Garsia, and this author among others, is seen to have a broader range of applicability.

The main result of the second part of the paper is the Betti number bound, Theorem 6.2, which enables us to deduce information about the minimal free resolution of the lexicographic ring from information about the underlying partially ordered set. Thus we can deduce that the Gorenstein property, as well as the Cohen–Macaulay property and sometimes even the type of the ring may be transferred from the partially ordered set.

We employ the following notation in this paper. We write $K$ for a field which is arbitrary but fixed throughout the paper, although in many cases we require only that $K$ be a Cohen–Macaulay (or possibly regular) ring. All rings considered in this paper are Noetherian (hence finitely generated) $K$-algebras, and we will often call them simply rings or algebras. We write $\mathbb{N}$ for the commutative semigroup of nonnegative integers, $\mathbb{Z}$ for the ring of all integers, and $\mathbb{Q}$ for the field of rational numbers. We write $[n]$ for the set $\{1, 2,..., n\}$. All simplicial complexes and partially ordered sets considered are finite, unless specified otherwise.

## 2. Combinatorial Decompositions of Rings

The idea of studying a $K$-algebra using "combinatorial decompositions" is a relatively recent innovation due to Baclawski and Garsia [7, 8, 20], although the use of combinatorial techniques in commutative algebra is much older, going back at least as far as Hilbert [22]. In this section we sketch the basic concepts and results, and refer the reader to [7, 8] for the details.

A $K$-algebra $R$ is said to be *$\mathbb{N}^m$-graded* if it can be written as a direct sum $R = \bigoplus_{\nu\in\mathbb{N}^m} R_\nu$ such that $R_0 = K$ and $R_\nu R_\mu \subseteq R_{\nu+\mu}$ for every $\nu, \mu \in \mathbb{N}^m$. The elements of $R_\nu$ are said to be homogeneous of *multidegree* $\nu$. This concept is extended to modules in a straightforward way. An $R$-module $M$ is said to be *$\mathbb{Z}^m$-graded* if it can be written as a direct sum $M = \bigoplus_{\nu\in\mathbb{Z}^m} M_\nu$ such that $M_\nu R_\mu \subseteq M_{\nu+\mu}$ for every $\nu, \mu \in \mathbb{Z}^m$. We will also write $\mathscr{H}_\nu M$ for $M_\nu$.

The case $m = 1$ is of particular importance, and every $\mathbb{N}^m$-graded $K$-algebra $R$ may be regarded as an ($\mathbb{N}$-) graded $K$-algebra by defining $R_n$ for $n \in \mathbb{N}$ to be $\bigoplus_{\|\nu\|=n} R_\nu$, where $\|\nu\| = c_1\nu_1 + \cdots + c_m\nu_m$, and where $c_1,..., c_m$ are positive integer constants. We call this an *associated graded algebra* of $R$. We distinguish the original $\mathbb{N}^m$-grading by calling it the *multigraded* structure of $R$.

The *Hilbert series* of a graded $K$-algebra $R$ is the power series $F_R(t) =$

$\sum_{n=0}^{\infty} \dim_K(R_n)\, t^n$. This series is known to be a rational function of $t$, and the order of the pole of $F_R(t)$ at $t=1$ is called the *Krull dimension* of $R$, abbreviated $K$-dim $R$. A *homogeneous system of parameters* for $R$ is a set of $r = K\text{-dim}\, R$ homogeneous elements $\theta_1,..., \theta_r$ of positive degree such that $R/(\theta_1,..., \theta_r)$ has Krull dimension zero, i.e., $R/(\theta_1,..., \theta_r)$ is finite dimensional as a vector space over $K$. A *frame* for $R$ is an *ordered* homogeneous system of parameters for $R$. Frames are known to exist for any $\mathbb{N}$-graded algebra, but there are $\mathbb{N}^2$-graded algebras that have no frames. The following is our basic combinatorial decomposition result:

THEOREM 2.1 (Baclawski and Garsia [7]). *Let $R$ be a finitely generated graded $K$-algebra of Krull dimension $r$. Then there is a frame $(\theta_1,..., \theta_r)$, a finite sequence of homogeneous elements $(\eta_1,..., \eta_N)$, and a function $k$: $[N] \to \{0, 1,..., r\}$ such that*

(1) *the images of $\eta_1,..., \eta_N$ in $R/(\theta_1,..., \theta_r)$ form a basis over $K$;*

(2) *every element of $R$ may be expressed in a unique fashion as a sum of the form*

$$\sum_{j=1}^{N} \eta_j p_j(\theta_1,..., \theta_{k(j)}),$$

*where $p_j$ is a polynomial in $K[X_1,..., X_{k(j)}]$;*

(3) *for every $j$, $\eta_j(\theta_{k(j)+1},..., \theta_r) \subseteq (\theta_1,..., \theta_{k(j)})$.*

We will say that a frame $(\theta_1,..., \theta_r)$ is a *Rees frame* if $\{\eta_1,..., \eta_N\}$ can be found so that property (2) of the above theorem is satisfied. We call $\{\eta_1,..., \eta_N\}$ a set of *separators* for the Rees frame $(\theta_1,..., \theta_r)$, and we call $k(j)$ the *level* of $\eta_j$. If in addition, $(\theta_1,..., \theta_r)$ and $\{\eta_1,..., \eta_N\}$ satisfy property (1), we say that the frame is *privileged.* If $R$ is a graded algebra for which there is a Rees frame all of whose separators have level $r$ (such a frame is said to be *basic*), then $R$ is said to be *Cohen–Macaulay* (abbreviated CM). In this case, one can show that every frame is a basic, *privileged* frame.

Suppose that the graded ring $R$ has been represented as a quotient $K[X]/I$ of a free polynomial ring $K[X] = K[X_i \mid 1 \leqslant i \leqslant n]$ by an ideal generated by homogeneous elements of $K[X]$. A *graded finite free resolution* of $R$ over $K[X]$ is an exact sequence of the form

$$0 \longrightarrow F_l \xrightarrow[\phi_l]{} F_{l-1} \xrightarrow[\phi_{l-1}]{} \cdots \xrightarrow[\phi_2]{} F_1 \xrightarrow[\phi_1]{} F_0 \xrightarrow[\phi_0]{} R \longrightarrow 0,$$

where each $F_p$ is a finite free $K[X]$-module whose generators are assigned degrees in such a way that each $\phi_p$ is a $K[X]$-homomorphism that preserves degree. One can show that there is a graded finite free resolution which simultaneously minimizes rank $(F_p)$ for all $p$ and that the resulting resolution is essentially unique. The number of generators of $F_p$ of degree $q \in \mathbb{Z}$ will be denoted $b_{p,q}(R)$ or simply $b_{p,q}$. The sum $b_p = \sum_{q \in \mathbb{Z}} b_{p,q} = \operatorname{rank}(F_p)$ is called the $p$th *Betti number* of $R$ over $K[X]$. One can also obtain the numbers $b_{p,q}(R)$ by using some homological algebra. Recall that $\operatorname{Tor}_p^{K[X]}(R, K)$ is isomorphic to the $p$th homology of the tensor product over $K[X]$ of any graded finite free resolution of $R$ with the $K[X]$-module $K = K[X]/(X_1,..., X_n)$. It is easy to see that $\operatorname{Tor}_p^{K[X]}(R, K)$ is a graded $K$-module, and that $b_{p,q}(R) = \dim_K \mathscr{H}_q \operatorname{Tor}_p^{K[X]}(R, K)$ [8, 9, 33].

The Hilbert Syzygy Theorem states that $b_p(R) = 0$ for $p > n$, where $n$ is the Krull dimension of $K[X]$. One can show that if $r$ is the Krull dimension of $R$, then $b_p \neq 0$ for $0 \leqslant p \leqslant n - r$. In other words, the length of a minimal free resolution of $R$ over $K[X]$ is at least $n - r$ and at most $n$. If the actual length is $l$, then the number $n - l$ is called the *depth* of $R$. One can show that depth $(R)$ is a property only of $R$ and not of the particular presentation of $R$ as a quotient $K[X]/I$, which we used to compute it. Indeed, depth $(R)$ is the length of any maximal *regular sequence* of $R$, where $\theta_1,..., \theta_k$ is said to be regular if each $\theta_i$ is homogeneous, of positive degree, and not a zero-divisor of $R/(\theta_1,..., \theta_{i-1})$. One can also use the depth to give another characterization of the CM property: the ring $R$ is CM if and only if depth$(R) = K$-dim$(R)$. We summarize these comments in the following:

PROPOSITION 2.2. *Let $R$ be a finitely generated graded $K$-algebra of Krull dimension $r$. Then the following are equivalent, where $d_i$ denotes* $\deg(\theta_i)$:

(1) *$R$ is Cohen–Macaulay*;

(2) *for some (every) frame $(\theta_1,..., \theta_r)$,*

$$F_R(t) = F_{R/(\theta_1,...,\theta_r)}(t) \Big/ \prod_{i=1}^{r} (1 - t^{d_i});$$

(3) *for some frame $(\theta_1,..., \theta_r)$ and some set $\{\eta_1,..., \eta_N\}$ of homogeneous elements of $R$, we have*

(a) *every element of $R$ may be written in the form*

$$\sum_{j=1}^{N} \eta_j p_j(\theta_1,..., \theta_r),$$

*where the $p_j$ are polynomials in $K[X_1,..., X_r]$ and*

(b) $$F_R(t) = \sum_j t^{\deg(\eta_j)} \Big/ \prod_{i=1}^{r} (1 - t^{d_i});$$

(4) *for some (every) frame* $(\theta_1,\ldots,\theta_r)$ *of* $R$, *the sequence* $\theta_1,\ldots,\theta_r$ *is a regular sequence of* $R$;

(5) *for some (every) representation of* $R$ *as a quotient* $K[X]/I$ *of a free polynomial algebra* $K[X]=K[X_1,\ldots,X_n]$ *by a homogeneous ideal,* $b_i(R)=0$ *for all* $i>n-r$ [8, 33].

## 3. Rings with Straightening Law

Let $V$ be a finite set, which we henceforth refer to as the *vertex set*. The (free) polynomial ring on a set of indeterminates $X_v$ in one-to-one correspondence with $V$ will be denoted $K[X_v \mid v \in V]$ or more succinctly as $K[V]$. The monomials in $K[V]$ form a partially ordered set (poset), under divisibility. This poset is isomorphic to the poset $\mathscr{M}(V)$ of multisets on elements of $V$ under inclusion. When $V$ is $[r]$ we will write $\mathscr{M}(r)$ for $\mathscr{M}([r])$. For a multiset $S \in \mathscr{M}(V)$, we write $X^S = \prod_{t\in S} X_t$ for the corresponding monomial of $K[V]$. Although $\mathscr{M}(V)$ consists only of multisets, we will sometimes abuse notation and refer to its elements as being the monomials $X^S$.

A *simplicial complex* $\Delta$ with vertices from $V$ is a nonempty collection of subsets of $V$ called *simplices* such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$. Note that the empty subset of $V$ is always in $\Delta$. A simplicial complex may be regarded as a poset, under inclusion; and we will write $P(\Delta)$ for the poset of *nonempty* simplices of $\Delta$. Conversely, if $P$ is a poset, then we may define a simplicial complex called the *order complex* (or *chain transform*) of $P$, denoted $\Delta(P)$, whose simplices are the chains (totally ordered subsets) of $P$. Starting with a simplicial complex $\Delta$, the simplicial complex $\Delta(P(\Delta))$ corresponds to the barycentric subdivision of the topological realization of $\Delta$.

For a monomial $X^S$ in $\mathscr{M}(V)$, the *support* of $S$ or of $X^S$ is the subset $\square(S)=\square(X^S)=\{v\in V \mid X^S \text{ is divisible by } X_v\}$. For a simplicial complex $\Delta$, we write $\mathscr{M}(\Delta)$ for the multisets $S$ whose support is in $\Delta$. The *Stanley–Reisner ring* of $\Delta$ denoted $K[\Delta]$ is then defined to be $K[V]/(X^T \notin \mathscr{M}(\Delta))$. It is easy to see that the images of the monomials in $\mathscr{M}(\Delta)$ form a basis of $K[\Delta]$. This is a special case of the following.

Definition 3.1. Let $R$ be a $K$-algebra. A *straightening law based on* $\Delta$ consists of a presentation of $R$ as a quotient ring $K[V]/I$ such that the images of $X^S$ for $S \in \mathscr{M}(\Delta)$ form a basis of $R$ over $K$. If $I$ is homogeneous with respect to some graded algebra structure on $K[V]$, then we say the straightening law is *graded*.

For such a ring $R$, if $T \in \mathscr{M}(V)\setminus\mathscr{M}(\Delta)$, then (the image of) $X^T$ in $R$ may be written uniquely as a linear combination of monomials $X^S$ in $\mathscr{M}(\Delta)$. We call this the *straightening formula* for $X^T$. The monomials $X^S$ in $\mathscr{M}(\Delta)$ are called *standard monomials*, while the others are *nonstandard*. Since the standard monomials form a vector space basis of $R$ and of $K[\Delta]$, there is a vector space isomorphism $\phi: K[\Delta] \to R$, which maps a standard monomial $X^S$ in $K[\Delta]$ to the corresponding standard monomial in $R$. In general, $\phi$ is not an isomorphism of rings, but it will be "close" to one in the sense defined in Theorem 4.2 below. The inverse map $\phi^{-1}: R \to K[\Delta]$ will be denoted $\psi$ and is called the *straightening algorithm* of $R$. The concept of a lexicographic ring was introduced independently by DeConcini *et al.* [13] and by Garsia [20]. However, our terminology differs from either of theirs.

The ideal $(X^T \notin \mathscr{M}(\Delta)) \subseteq K[V]$ that defines the Stanley–Reisner ring is generated by the monomials $X^T$ for which $T \subseteq V$ is a minimal non-simplex of $\Delta$; i.e., $T \notin \Delta$ but every proper subset of $T$ is in $\Delta$. When $\Delta$ is of the form $\Delta(P)$, for some poset $P$, then the minimal non-simplices of $\Delta(P)$ all have cardinality 2; hence, the ideal defining $K[\Delta(P)]$ is generated by the quadratic monomials $X_v X_w$, where $v$ and $w$ are incomparable in $P$. We now show that every straightening law is equivalent to one based on a simplicial complex of the form $\Delta(P)$.

PROPOSITION 3.2. *A ring with a straightening law based on the simplicial complex $\Delta$ is isomorphic to a ring with a straightening law based on $\Delta(P(\Delta))$.*

*Proof.* Let $R$ be a ring with a straightening law based on $\Delta$. By definition $R$ is a quotient $K[V]/I$. Write $P$ for the poset $P(\Delta)$. Now the monomials of $K[X] = K[X_a \mid a \in P]$ are of the form $X_{\sigma_1} \cdots X_{\sigma_n}$, where $\sigma_1, ..., \sigma_n$ are (nonempty) simplices of $\Delta$. We define a $K$-algebra homomorphism $K[X] \to K[V]$ by mapping $X_\sigma$ to $X^\sigma$, where $\sigma \in P$ and $X^\sigma = \prod_{v \in \sigma} X_v$. Write $f: K[X] \to R$ for the composition $K[X] \to K[V] \to K[V]/I = R$. It is easy to see that $f$ is surjective, since the minimal elements $\{v\}$ of $P$ generate $R$ as a $K$-algebra and $X_{\{v\}}$ in $K[X]$ gets mapped to $X_v$ in $R$. Therefore $R$ is isomorphic to $K[X]/\mathrm{Ker}(f)$. We wish to show that the monomials in $\mathscr{M}(\Delta(P))$ form a basis of $K[X]/\mathrm{Ker}(f)$.

A monomial $X^S$ of $\mathscr{M}(\Delta(P))$ may be written in the form $X_{\sigma_1} X_{\sigma_2} \cdots X_{\sigma_n}$, where $\sigma_1 \supseteq \sigma_2 \supseteq \cdots \supseteq \sigma_n$ are nonempty simplices of $\Delta$. The image of such a monomial is $X^{\sigma_1} X^{\sigma_2} \cdots X^{\sigma_n}$ in $R$.

On the other hand, let $X^S$ be a monomial of $\mathscr{M}(\Delta)$. By definition these form a basis of $R$. Now define simplices $\sigma_1, \sigma_2, ...,$ inductively as follows. Let $\sigma_1$ be $\square(X^S)$ and let $X^{S_2} = X^S / X^{\sigma_1}$. Then define $\sigma_2$ to be $\square(X^{S_2})$ and so on. If we continue this process, it eventually stops say with $\sigma_n$; and the $\sigma_i$'s satisfy $\sigma_1 \supseteq \sigma_2 \supseteq \cdots \supseteq \sigma_n$ and $X^S = X^{\sigma_1} X^{\sigma_2} \cdots X^{\sigma_n}$. Conversely, if $X^S$ may be written in the form $X^{\tau_1} X^{\tau_2} \cdots X^{\tau_m}$ such that $\tau_1 \supseteq \tau_2 \supseteq \cdots \supseteq \tau_m \neq \phi$ and $\tau_i \in \Delta$ for all $i$, then it is easy to see that $n = m$ and $\sigma_i = \tau_i$ for all $i$. Thus every monomial of $\mathscr{M}(\Delta)$ is uniquely expressible as the image of a monomial of $\mathscr{M}(\Delta(P))$. The result then follows. ∎

As a result of Proposition 3.2, it suffices to consider rings with a straightening law based on simplicial complexes of the form $\Delta(P)$, where $P$ is a poset, and we will henceforth do so. Because of this we will abbreviate $K[\Delta(P)]$ to simply $K[P]$. We will also write $K[X]$ for $K[X_a \mid a \in P]$, relying on the context to indicate which poset is intended. The ring $K[P]$ has a natural multigrading and a natural frame with respect to this structure. Before discussing these we pause to discuss some terminology from the theory of partially ordered sets.

For a poset $P$, we write $\hat{P}$ for the poset obtained by adjoining two new elements $\hat{0}$ and $\hat{1}$ to $P$ such that $\hat{0} < x < \hat{1}$ for all $x \in P$. The cardinality of the longest chain of $P$ is called the *rank* of $P$, denoted $r(P)$. We then extend this notion to elements of $P$ by defining the *rank* of $x \in P$ to be $r(\hat{0}, x]$. The elements of $P$ having rank $l$, where $1 \leqslant l \leqslant r(P)$, form an antichain $P_l$. The elements of $P_1$ are the minimal elements of $P$, those of $P_2$ are the minimal elements of $P \backslash P_1$ and so on. When $P = P(\Delta)$ for a simplicial complex $\Delta$, the rank of $\sigma \in P$ is the same as its cardinality as a subset of the vertex set $V$.

*Remark* 3.3. The rank function on a poset $P$ is a special case of a "coloring" of a simplicial complex. Let $\Delta$ be a simplicial complex of rank $r$ on a vertex set $V$. A *coloring* of $\Delta$ is a function $c\colon V \to [r]$ such that for all $\sigma \in \Delta$, $|c(\sigma)| = |\sigma|$, i.e., the colors assigned to the vertices of $\sigma$ are distinct. Most of our definitions and results on simplicial complexes of the form $\Delta(P)$ extend to colorable complexes. A pure, colorable complex is also called a *completely balanced complex* [6, 36].

*Remark* 3.4. Let $P$ be a poset of rank $r$. Write $e_i \in \mathbb{N}^r$ for the $i$th standard basis element of $\mathbb{N}^r$. The *natural multigrading* of $K[P]$ is the $\mathbb{N}^r$-grading defined by $\deg(X_a) = e_{r(a)}$ for every $a \in P$. We will use only this multigrading on $K[P]$ and gradings associated to it. For variety of notation we will often regard a multidegree in $\mathbb{N}^r$ as a multiset in $\mathscr{M}(r) = \mathscr{M}([r])$ in the obvious way; i.e., the multiplicity of $i \in [r]$ in $v \in \mathbb{N}^r$ is $v_i$; and we will refer to the elements of $\mathscr{M}(r)$ as *shapes*.

In $K[P]$ we define $\theta_i$ to be the homogeneous element $\sum_{a \in P, r(a) = i} X_a$. One can show that $(\theta_1, \ldots, \theta_r)$ is a frame for the multigraded algebra $K[P]$. This frame is natural in many ways. For some of its nice features see Baclawski and Garsia [7].

By means of the Stanley–Reisner ring $K[P]$ we may transfer ring-theoretic notions to partially ordered sets. For example, a poset $P$ is said to be *Cohen–Macaulay* if $K[P]$ is so. A poset $P$ is said to be *almost Cohen–Macaulay* if every open interval $(x, y)$ of $\hat{P}$ is CM, except possibly for $(\hat{0}, \hat{1})$. [3, 36] The original definitions of CM and ACM posets used concepts from algebraic topology. For early work in this area see Folkman [19] and Baclawski [1, 2]. It was Reisner [31] who (independently) showed that the topological and ring-theoretic concepts coincided. For more recent treatments of the theory of CM posets see [3, 7, 20, 36]. For the theory of CM *complexes* see also [24, 34].

As a result of the theory of CM posets as developed in the references above, we now have many tools for showing that a poset is CM. The principal example which we mention for later use is the following. A lattice $L$ is said to be *(upper) semimodular* if $x, y \in L$ both covering $x \wedge y$ implies that $x \vee y$ covers both $x$ and $y$. If $L$ is semimodular, then $L$ is CM. For a proof using an explicit combinatorial decomposition of $K[L]$ see Garsia [20]. The result has also been generalized in various directions. See [3] for some of these.

## 4. Lexicographic Conditions and the Transfer Theorem

In this section we introduce the concept of an admissible (or "lexicographic") straightening law. Roughly speaking, a straightening algorithm is admissible if the straightening formula for a given nonstandard monomial is a linear combination of "earlier" monomials. To make the concept of "earlier" monomial precise, we need to formalize the concept of an admissible partial order on monomials. As we will then see, there exist several examples of admissible partial orders, typically defined by some kind of lexicographic order. Our main result is that if a ring $R$ has an admissible straightening law, then we can "transfer" Rees frames from $K[P]$ to $R$. This "Transfer Theorem" has several immediate consequences, including DeConcini's conjecture.

Recall that the multidegree of a homogeneous element of $K[P]$ may be regarded as an element of $\mathscr{M}(r)$, where $r = r(P)$. The multidegree (shape) of a monomial $w$ is denoted $\lambda(w)$. The set of shapes $\mathscr{M}(r) = \mathbb{N}^r$ has a natural commutative semigroup structure given by addition in $\mathbb{N}^r$. We say that a partial order $\leqslant$ on $\mathscr{M}(r)$ is *admissible* if it satisfies

(1) if $\lambda \leqslant \nu$ and $\lambda' < \nu'$ then $\lambda + \lambda' < \nu + \nu'$;

(2) for any $\lambda \in \mathscr{M}(r)$, $\{\nu \mid \nu \leqslant \lambda\}$ is finite.

Both conditions are actually stronger than we need, but this definition suffices for all the examples we have in mind.

Suppose now that we choose one of the associated gradings of $K[P]$. This is equivalent to choosing an additive map $\|\cdot\|: \mathscr{M}(r) \to \mathbb{N}$, where $\|\lambda\|$ is the degree of any monomial having shape $\lambda$. In the special case for which all the indeterminates of $K[X]$ are given degree 1, then $\|\lambda\|$ reduces to the cardinality $|\lambda|$ of the shape $\lambda$ as a multiset. We say that an admissible order

$\leqslant$ is *compatible with a grading* on $K[P]$ if for every $\lambda$, $\nu \in \mathscr{M}(r)$, $\lambda < \nu$ implies $\|\lambda\| \leqslant \|\nu\|$ and $\|\lambda\| < \|\nu\|$ implies $\lambda < \nu$. In other words, the admissible order is the ordinal sum of the sequence of partially ordered subsets $\{\lambda|\, \|\lambda\| = n\}$.

We now give some examples of admissible orders. The proof that these are admissible is given later. All these are most naturally expressed by writing a shape $\lambda$ either in the form of a nonincreasing sequence $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$ or in the form of a nondecreasing sequence $\lambda^1 \leqslant \lambda^2 \leqslant \cdots \leqslant \lambda^k$, where $k = |\lambda|$ and $\lambda = \{\lambda_1,..., \lambda_k\} = \{\lambda^1,..., \lambda^k\}$. We use the conventions that $\lambda_i = 0$ and $\lambda^i = r+1$ for $i > k$. Our first example is the *(ordinary) lexicographic order*. We say that $\lambda$ *strictly precedes* $\nu$, written $\lambda <_{\mathrm{L}} \nu$, if either $\|\lambda\| < \|\mu\|$ or $\|\lambda\| = \|\mu\|$ and there is an index $l$ such that $\lambda_i = \nu_i$ for $i < l$ and $\lambda_l < \nu_l$. To obtain a second example, simply replace $\lambda_i$ by $\lambda^i$ in the definition above. We write $<_{\mathrm{L}'}$ for this order. Our third partial order is *reverse lexicographic order*. In this case we say that $\lambda$ strictly precedes $\nu$, written $\lambda <_{\mathrm{R}} \nu$, if either $\|\lambda\| < \|\mu\|$ or $\|\lambda\| = \|\mu\|$ and there is an index $l$ such that $\lambda_i = \nu_i$ for $i > l$ and $\lambda_l < \nu_l$. As with the ordinary lexicographic order, we can modify this order by replacing $\lambda_i$ by $\lambda^i$ in the definition, obtaining a partial order $<_{\mathrm{R}'}$. Another order that is frequently encountered is *dominance*. We say that $\lambda$ *dominates* $\nu$, and write $\lambda \geqslant_{\mathrm{D}} \nu$, if either $\|\lambda\| > \|\nu\|$ or $\|\lambda\| = \|\nu\|$ and for every $l$, $\lambda_1 + \lambda_2 + \cdots + \lambda_l \geqslant \nu_1 + \nu_2 + \cdots + \nu_l$. Since $\lambda \geqslant_{\mathrm{D}} \nu$ implies that $\lambda \geqslant_{\mathrm{L}} \nu$, we lose no generality in the context of this paper if we consider only the various lexicographic orders. To generate still more examples, we note that if $\leqslant$ is admissible and compatible with the grading, then if we reverse the order on each subset $\{\lambda|\, \|\lambda\| = n\}$, the result is another admissible order, compatible with the grading, since $\|\lambda + \nu\| = \|\lambda\| + \|\nu\|$. We now prove that the lexicographic orders defined above are all admissible.

**PROPOSITION 4.1.** *The partial orders $\leqslant_{\mathrm{L}}$, $\leqslant_{\mathrm{L}'}$, $\leqslant_{\mathrm{R}}$ and $\leqslant_{\mathrm{R}'}$ on $\mathscr{M}(r)$ are admissible.*

*Proof.* The second part of the definition of an admissible order trivially holds for any order compatible with a grading since $\{\lambda \mid \|\lambda\| = n\}$ is finite for any $n$. Thus to show admissibility we need only show that if $\|\lambda\| = \|\nu\|$, $\|\lambda'\| = \|\nu'\|$, $\lambda \leqslant \nu$ and $\lambda' < \nu'$, then $\lambda + \lambda' < \nu + \nu'$. We will only prove that $\leqslant_{\mathrm{L}}$ is admissible, since the other cases are so similar. First consider the case $\lambda = \nu$. To show that $\lambda + \lambda' < \lambda + \nu'$, we may use induction on $|\lambda|$ and hence we may assume that $|\lambda| = 1$. In this special case, the result easily follows by examining these three cases: (1) $\lambda_1 \geqslant \nu'_l$, (2) $\nu'_l > \lambda_1 \geqslant \lambda'_l$ and (3) $\lambda'_l > \lambda_1$, where $l$ is the first index such that $\lambda'_l \neq \nu'_l$. Now consider the case $\lambda < \nu$. Let $m$ be the first index such that $\lambda_m \neq \nu_m$. Let $\alpha = (\lambda_1 \geqslant \cdots \geqslant \lambda_{m-1})$, $\beta = (\lambda_m \geqslant \cdots \geqslant \lambda_{|\lambda|})$ and $\gamma = (\nu_m \geqslant \cdots \geqslant \nu_{|\nu|})$. Then $\lambda = \alpha + \beta$ and $\nu = \alpha + \gamma$. Decompose $\lambda'$ and $\nu'$ in a similar manner, so that $\lambda' = \alpha' + \beta'$ and $\nu' =$

$\alpha' + \gamma'$. Since $\beta_1 < \gamma_1$ and $\beta'_1 < \gamma'_1$, it is trivial that $\beta + \beta' < \gamma + \gamma'$. By an earlier observation, we have that $\alpha + \alpha' + \beta + \beta' < \alpha + \alpha' + \gamma + \gamma'$, but this inequality is the same as $\lambda + \lambda' < \mu + \mu'$. ∎

DEFINITION 4.2. Let $R$ be a $K$-algebra and $P$ a poset of rank $r$. A *lexicographic* (or *admissible*) *straightening law on* $R$ *based on* $P$ consists of

(1) a straightening law on $R$ based on $\Delta(P)$,

(2) an admissible order $\leqslant$ on $\mathscr{M}(r)$ such that for every nonstandard monomial $w$ of $K[X]$, the straightening formula for $w$ is a linear combination of monomials whose shapes strictly precede $\lambda(w)$.

A *graded lexicographic straightening law based on* $P$ is a lexicographic straightening law together with an associated grading of $K[P]$ with respect to which the straightening law is graded and the admissible order is compatible. A *graded lexicographic ring based on* $P$ is a ring with a (graded) lexicographic straightening law based on $P$.

DeConcini *et al.* [13] consider a stronger requirement than we require for a ring to be lexicographic. More precisely, let $P$ be a poset of rank $r$, and let $\{a_1, a_2, \ldots, a_n\}$ be a total ordering of $P$ such that $r(a_1) \leqslant r(a_2) \leqslant \cdots \leqslant r(a_n)$. Each monomial of $K[X]$ may then be written uniquely in the form $X_{b_1} \cdots X_{b_m}$, where $b_1 \leqslant \cdots \leqslant b_m$ in the total order on $P$. Let $w = X_{b_1} \cdots X_{b_m}$ and $v = X_{c_1} \cdots X_{c_l}$ be two such monomials. We say $w$ *strongly precedes* $v$ if there is an index $j$ such that $b_i = c_i$ for $i \leqslant j$ and either $m = j < l$ or $b_{j+1} < c_{j+1}$ in the usual partial order on $P$. We claim that if $w$ strongly precedes $v$ as above, then $\lambda(w) <_{\mathrm{L}'} \lambda(v)$. To see this, note that in the case $m = j < l$ we have $\|\lambda(w)\| < \|\lambda(v)\|$, while if $b_i = c_i$ for $i \leqslant j$ and $b_{j+1} < c_{j+1}$, then since $\lambda(w)^i = r(b_i)$ and $\lambda(v)^i = r(c_i)$, we have that $\lambda(w) <_{\mathrm{L}'} \lambda(v)$. We will say that a lexicographic ring with respect to the $<_{\mathrm{L}'}$ order is *strongly lexicographic* if every nonstandard monomial $w$ is a linear combination of monomials that strongly precede $w$.

Let $R$ be a lexicographic ring based on $P$. Then the straightening formula for a nonstandard monomial $w$ may be found as follows. Write $w$ as a product $X_{t_1} X_{t_2} \cdots X_{t_n}$, where $r(t_1) \geqslant r(t_2) \geqslant \cdots \geqslant r(t_n)$. A *violation* of standardness is a quadratic factor $X_{t_i} X_{t_{i+1}}$ such that $t_i$ and $t_{i+1}$ are not comparable in $P$. To straighten $w$ we find the first violation $X_{t_i} X_{t_{i+1}}$, i.e., the one for which $i$ is smallest. Now straighten $X_{t_i} X_{t_{i+1}}$, obtaining a sum $\sum_j c_j v_j$, where every $\lambda(v_j)$ precedes $\lambda(X_{t_i}) + \lambda(X_{t_{i+1}})$. It is an immediate consequence of admissibility, that the shape of every monomial in the right-hand side of $w = \sum_j c_j X_{t_1} \cdots X_{t_{i-1}} v_j X_{t_{i+2}} \cdots X_{t_n}$ precedes $\lambda(w)$. Thus we can straighten a nonstandard monomial $w$ by finding the first violation, straightening it, then looking for the first violation in each of the resulting terms and so on. By admissibility, this procedure eventually terminates with the straightening formula for $w$.

We now state the main result of this section.

TRANSFER THEOREM 4.3. *Let $P$ be a poset of rank $r$, and let $R$ be a lexicographic ring based on $P$. Suppose that $(\theta_1,\ldots,\theta_r)$ is a sequence, and $\mathscr{S}$ is a set, of homogeneous elements of the multigraded algebra $K[P]$. Suppose also that $k\colon \mathscr{S} \to \{0, 1,\ldots, r\}$ is a fraction. Then,*

(1) *if $\{\eta\theta^I \mid \eta \in \mathscr{S}, I \in \mathscr{M}(k(\eta))\}$ spans $K[P]$ as a vector space, then $\{\phi(\eta)\,\phi(\theta)^I \mid \eta \in \mathscr{S}, I \in \mathscr{M}(k(\eta))\}$ spans $R$;*

(2) *if $\{\eta\theta^I \mid \eta \in \mathscr{S}, I \in \mathscr{M}(k(\eta))\}$ is a basis of $K[P]$, then $\{\phi(\eta)\,\phi(\theta)^I \mid \eta \in \mathscr{S}, I \in \mathscr{M}(k(\eta))\}$ is a basis of $R$.*

In effect, the theorem above is saying that the vector space isomorphism $\phi$ may be used to "transfer" frames, Rees frames and sets of separators from $K[P]$ to $R$. This process is only one-way: $\psi$ does not preserve Rees frames, in general (see [7, Example 4.4]). However, we show in Theorem 4.7 that certain Rees frames can be transferred from $R$ to $K[P]$.

The proof of Theorem 4.3 depends on the following lemma which expresses more clearly the sense in which $\phi$ is a "perturbation" of a ring isomorphism:

STRAIGHTENING LEMMA 4.4. *Let $f_1,\ldots, f_k$ be homogeneous polynomials in the $\mathbb{N}^r$-graded algebra $K[P]$. Then $\phi(f_1 f_2 \cdots f_k) - \phi(f_1)\,\phi(f_2) \cdots \phi(f_k)$ is a linear combination of monomials whose shapes strictly precede the sum of the shapes of $f_1, f_2,\ldots, f_k$.*

*Proof.* By the linearity of $\phi$, we may assume that each $f_i$ is a monomial. In this case $f_1 \cdots f_k$ is either a monomial or zero depending on whether $\square(f_1) \cup \cdots \cup \square(f_k)$ is in $\Delta(P)$ or not. In the former case, $\phi(f_1 \cdots f_k) = \phi(f_1) \cdots \phi(f_k)$. In the latter case, we have, by definition of a lexicographic ring, that $\phi(f_1) \cdots \phi(f_k)$ is a linear combination of monomials whose shapes strictly precede $\lambda(f_1) + \cdots + \lambda(f_k)$. Since $\phi(f_1 \cdots f_k) = 0$ in this case, we are finished. ∎

*Proof of Theorem* 4.3. We first show the proof of part (1). Let $w$ be a monomial of $R$. By assumption we may write $\psi(w)$ as a linear combination $\psi(w) = \sum c_{\eta,I}\eta\theta^I$ of elements of $\{\eta\theta^I \mid \eta \in \mathscr{S}, I \in \mathscr{M}(k(\eta))\}$. Since $\psi(w)$, the $\theta_i$'s and all elements of $\mathscr{S}$ are homogeneous, we may assume, by taking the homogeneous component of shape $\lambda(\psi(w))$, that all terms in this linear combination have the same shape as $\psi(w)$. By Lemma 4.4, we have that

$$w - \sum_{\eta,I} c_{\eta,I}\phi(\eta)\,\phi(\theta)^I$$

is a linear combination of monomials whose shapes strictly precede $\lambda(w)$. Since only finitely many shapes precede $\lambda(w)$, we may repeat this argument to each term in the difference above and use induction to conclude that $w$ may be written as a linear combination of elements of $\{\phi(\eta)\,\phi(\theta)^I \mid \eta \in \mathscr{S}, I \in \mathscr{M}(k(\eta))\}$. Thus part (1) follows.

We now examine part (2). We wish to show that $\{\phi(\eta)\,\phi(\theta)^I \mid \eta \in \mathscr{S}, I \in \mathscr{M}(k(\eta))\}$ is a basis of $R$. Let $D \subseteq \mathscr{M}(r)$ be a finite order-ideal. Write $W(D)$ for the subspace of $K[P]$ spanned by monomials of shape $\lambda \in D$. The hypothesis of part (2) immediately implies that $\{\eta\theta^I \mid \lambda(\eta\theta^I) \in D\}$ is a basis of $W(D)$. Now by Lemma 4.4, if we expand $\phi(\eta)\,\phi(\theta)^I$ as a linear combination of monomials, we will get a linear combination of monomials whose shapes precede or equal $\lambda(\eta\theta^I)$. Thus if $\lambda(\eta\theta^I) \in D$, then all the monomials occurring in the expansion of $\phi(\eta)\,\phi(\theta)^I$ will have shapes in $D$. If we examine the proof of part (1) we find that we also showed that for any finite order-ideal $D \subseteq \mathscr{M}(r)$, $\{\phi(\eta)\,\phi(\theta)^I \mid \eta \in \mathscr{S}, I \in \mathscr{M}(k(\eta)), \lambda(\eta\theta^I) \in D\}$ spans $\phi(W(D))$. Since $W(D)$ and $\phi(W(D))$ have the same dimension over $K$, this set is a basis of $\phi(W(D))$. In particular, it is linearly independent in $\phi(W(D))$. Now suppose that $\sum_{\eta,I} c_{\eta,I}\phi(\eta)\,\phi(\theta)^I = 0$ were a dependence relation for $\{\phi(\eta)\,\phi(\theta)^I\}$. Since such a relation has only finitely many terms, only finitely many shapes occur. By admissibility, these shapes are all in some finite order-ideal $D$ in $\mathscr{M}(r)$, giving us a dependence relation in $W(D)$. We thus have a contradiction, and hence $\{\phi(\eta)\,\phi(\theta)^I \mid \eta \in \mathscr{S}, I \in \mathscr{M}(k(\eta))\}$ is linearly independent in $R$ over $K$. Since this set already spans by part (1), we conclude that it is a basis. ∎

We now give some important consequences of the Transfer Theorem.

THEOREM 4.5. *Let $R$ be a graded lexicographic ring based on $P$. Suppose that $(\theta_1,..., \theta_d)$ is a homogeneous regular sequence in the multigraded algebra $K[P]$. Then the corresponding sequence $(\phi(\theta_1),..., \phi(\theta_d))$ is regular in $R$.*

*Proof.* We use Stanley's characterization of regular sequences [35, Corollary 3.2]. A sequence $(\tau_1,..., \tau_d)$ of homogeneous elements of positive degree in a graded ring $R$ form a regular sequence if and only if

$$F_R(t) = F_{R/(\tau_1,\ldots,\tau_d)}(t) \Big/ \prod_{i=1}^{d} (1 - t^{\deg(\tau_i)}).$$

Choose a set $\mathscr{S}$ of homogeneous elements of $K[P]$ whose images in $K[P]/(\theta_1,..., \theta_d)$ form a basis over $K$. Since each $\theta_i$ is homogeneous of positive degree, we can use induction to show that $\{\eta\theta^I \mid \eta \in \mathscr{S}, I \in \mathscr{M}(d)\}$ spans $K[P]$. By the regularity of $(\theta_1,..., \theta_d)$ and Stanley's theorem, $\{\eta\theta^I\}$ is a basis of $K[P]$. By part (2) of the Transfer Theorem, $\{\phi(\eta)\,\phi(\theta)^I \mid \eta \in \mathscr{S},$

$I \in \mathcal{M}(d)\}$ is a basis of $R$ over $K$. Hence $F_R(t) = F_{\mathcal{S}}(t) \prod_{i=1}^{d} (1 - t^{\deg(\theta_i)})^{-1}$, where $F_{\mathcal{S}}(t) = \sum_{\eta \in \mathcal{S}} t^{\deg(\eta)}$. Since $\{\phi(\eta)\, \phi(\theta)^I\}$ is a basis of $R$, it follows that the images of $\{\phi(\eta) \mid \eta \in \mathcal{S}\}$ span $R/(\phi(\theta_1),..., \phi(\theta_d))$. Choose a subset $\mathcal{T} \subseteq \mathcal{S}$ such that the images of $\{\phi(\eta) \mid \eta \in \mathcal{T}\}$ form a basis. By induction on degree we have that $\{\phi(\eta)\, \phi(\theta)^I \mid \eta \in \mathcal{T}, I \in \mathcal{M}(d)\}$ spans $R$, since every $\phi(\theta_i)$ is homogeneous of positive degree. But $\{\phi(\eta)\, \phi(\theta)^I \mid \eta \in \mathcal{S}, I \in \mathcal{M}(d)\}$ is a basis of $R$. Therefore $\mathcal{T} = \mathcal{S}$, and hence $F_{\mathcal{S}}(t) = F_{R/(\phi(\theta_1),..., \phi(\theta_d))}(t)$. Combining this with our earlier result we have that

$$F_R(t) = R_{R/(\phi(\theta_1),..., \phi(\theta_d))}(t) \Big/ \prod_{i=1}^{d} (1 - t^{\deg(\theta_i)}),$$

which is equivalent to the regularity of $(\phi(\theta_1),..., \phi(\theta_d))$, by Stanley's theorem. ∎

The following corollary may seem to be an immediate consequence of the theorem above, but one should remember that a *multi*graded algebra of positive depth need not have *any* nonconstant homogeneous nonzero-divisors.

COROLLARY 4.6. *Let $R$ be a graded lexicographic ring based on $P$. If $P$ is ACM, then* $\operatorname{depth}(R) \geqslant \operatorname{depth} K[P]$. *In particular, if $P$ is CM, then $R$ is CM.*

*Proof.* Let $(\theta_1,..., \theta_r)$ be the natural homogeneous frame defined in Remark 3.4. By [5, Corollary 12], if $d = \operatorname{depth} K[P]$, then any sequence of $d$ elements of $\{\theta_1,..., \theta_r\}$ is a homogeneous regular sequence of the multigraded ring $K[P]$. By Theorem 4.5, $\operatorname{depth}(R) \geqslant d$. In particular if $P$ is CM, then $(\theta_1,..., \theta_r)$ is a regular sequence by Proposition 2.2(4) (we do not require [5, Corollary 12] in this case). By Theorem 4.5, $\operatorname{depth}(R) = d$. By Proposition 2.2(4) again, $R$ is CM. ∎

The second part of Corollary 4.6 for the special case of a strongly lexicographic ring is DeConcini's conjecture.

We now consider a partial converse to Theorem 4.3. To state this we need some new notation. Suppose that $\theta_1,..., \theta_r$ and $\eta$ are homogeneous elements of the $\mathbb{N}^r$-graded algebra $K[P]$ as in Theorem 4.3. Let $\sigma_i = \phi(\theta_i)$ and $\xi = \phi(\eta)$. Now for $S \in \mathcal{M}([r])$, the element $\psi(\xi\sigma^S)$ of $K[P]$ will not, in general, be homogeneous in $K[P]$. Thus while we cannot assign a shape to $\xi\sigma^S$ as an element of $R$, we can nevertheless formally assign the shape $\lambda(\eta\theta^S)$ to $\xi\sigma^S$. In effect, we assign the shape $\lambda(\eta\theta^S)$ to the *factorization* of $\xi\sigma^S$ as a product of $\xi$ and $\sigma_i$'s.

THEOREM 4.7. *Let $R$ be a lexicographic ring based on a poset $P$ of rank $r$. Suppose that $(\sigma_1,..., \sigma_r)$ is a Rees frame for $R$ with separators $(\xi_1,..., \xi_N)$. Suppose that*

(1) *for every $i$ and $j$, $\psi(\sigma_i)$ and $\psi(\xi_j)$ are homogeneous in $K[P]$;*

(2) *for every monomial $w$ in $R$, whose Rees expansion is given by $w = \sum_{j=1}^{N} \sum_{S \in \mathscr{M}(k(j))} c_{j,S} \xi_j \sigma^S$, we have that $\lambda(\xi_j \sigma^S)$ precedes or is equal to $\lambda(w)$ whenever $c_{j,S} \neq 0$.*

*Then $(\psi(\sigma_1),..., \psi(\sigma_r))$ is a Rees frame for $K[P]$ with separators $(\psi(\xi_1),..., \psi(\xi_N))$. Moreover, if $R$ is graded, then $(\sigma_1,..., \sigma_r)$ is a basic privileged frame if and only if $(\psi(\sigma_1),..., \psi(\sigma_r))$ is also a basic privileged frame. In particular, $R$ is CM if and only if $K[P]$ is CM.*

*Proof.* Write $\eta_j$ for $\psi(\xi_j)$ and $\theta_i$ for $\psi(\sigma_i)$ for all $i, j$. We will first show that $\{\eta_j \theta^S \mid j \in [N],\ S \in \mathscr{M}(k(j))\}$ is a linearly independent set in $K[P]$. Suppose not. Since all the $\eta_j \theta^S$'s are homogeneous in $K[P]$ by hypothesis (1), there is a linear combination

$$\sum_{j,S} c_{j,S} \eta_j \theta^S$$

that vanishes in $K[P]$ and all terms appearing with a nonzero coefficient have the same shape $\lambda$. Consider the corresponding expression $\sum_{j,S} c_{j,S} \xi_j \sigma^S$ in $R$. By the Straightening Lemma 4.4 and hypothesis (1), when we expand this expression as a linear combination $\sum_w d_w w$ of monomials $w$ in $R$, the monomials which appear have shapes that (strictly) precede $\lambda$. Each such monomial $w$ has a Rees expansion

$$w = \sum_{j,S} b_{j,S}(w)\, \xi_j \sigma^S.$$

By hypothesis (2), if $b_{j,S}(w) \neq 0$, then $\lambda(\xi_j \sigma^S) \leqslant \lambda(w)$. Since $\lambda(w) < \lambda$, we have that all such $\xi_j \sigma^S$ satisfy $\lambda(\xi_j \sigma^S) < \lambda$. Now

$$\begin{aligned} \sum_{j,S} c_{j,S} \xi_j \sigma^S &= \sum_w d_w w = \sum_w d_w \sum_{j,S} b_{j,S}(w)\, \xi_j \sigma^S \\ &= \sum_{j,S} \left( \sum_w d_w b_{j,S}(w) \right) \xi_j \sigma^S. \end{aligned}$$

Hence $c_{j,S} = \sum_w d_w b_{j,S}(w)$ for all $j$, $S$. But every $\xi_j \sigma^S$ with $c_{j,S} \neq 0$ has shape $\lambda$ and no such term appears on the right-hand side above. Thus $c_{j,S} = 0$ for all $j$, $S$; and we conclude that $\{\eta_j \theta^S\}$ is a linearly independent set in $K[P]$.

It remains to prove that $\{\eta_j \theta^S\}$ spans $K[P]$ as a vector space. To this end we simply reverse the argument in the second half of the proof of Theorem 4.3. Using the notation introduced there, we know that for any finite order-ideal $D \subseteq \mathscr{M}(r)$, the elements $\eta_j \theta^S$ having shape $\lambda \in D$ are linearly independent in $W(D)$. We also know that the elements $\xi_j \sigma^S$ having shape $\lambda \in D$ form a basis of $\phi(W(D))$. Since $\dim_K W(D) = \dim_K \phi(W(D))$, we conclude that $\{\eta_j \theta^S\}$ spans any $W(D)$ and hence all of $K[P]$. ∎

Theorems 4.3 and 4.7 suggest that there ought to be a sense in which one $K$-algebra can "dominate" another. The proper definition has yet to be found. But it should be such that if $R$ dominates $R'$ then a Rees frame can be transferred from $R$ to $R'$. In Theorem 4.3, $K[P]$ dominates a lexicographic ring based on $P$. In Theorem 4.7, $K[P]$ and $R$ dominate each other. We will discuss these ideas in a sequel to this paper.

We end this section with an unpublished result of DeConcini. The special case of this result for which the ring is strongly lexicographic is proved in DeConcini *et al.* [13]. The proof given there, however, does not appear to generalize.

THEOREM 4.8 (DeConcini). *If $R$ is a lexicographic ring with respect to an admissible total order, then $R$ is reduced; i.e., for any $x \in R$ and $n > 1$, $x^n = 0$ implies that $x = 0$.*

We remark that all the lexicographic orders mentioned in Proposition 4.1 are total orders. In particular, by the remark following Definition 4.2, the theorem above applies to strongly lexicographic rings.

*Proof.* Let $R$ be a lexicographic ring based on $P$, whose admissible order is total. It suffices to show that $x^2 = 0$ implies that $x = 0$. Suppose that $x^2 = 0$ but that $x \neq 0$. Let $y \in R$ be the nonzero homogeneous component of $x$ of highest shape. Let $\lambda = \lambda(y)$, and let $\psi(y) = \sum a_I X^I$ in $K[P]$. Then $(\psi(y))^2 = \sum (a_I)^2 (X^I)^2$. Thus $(\psi(y))^2 \neq 0$. Now every term of $x - y$ has shape preceding $\lambda$. Thus by admissibility, $x^2 - y^2 = (x-y)^2 + 2y(x-y)$ is a linear combination of monomials whose shapes precede $2\lambda$. By the Straightening Lemma 4.4, the same can be said of $x^2 - \phi((\psi(y))^2)$ and hence of $\psi(x^2) - (\psi(y))^2$. Now $x^2 = 0$, and $(\psi(y))^2$ is homogeneous of shape $2\lambda$. Thus $(\psi(y))^2 = 0$, and we have a contradiction. It follows that $R$ is reduced. ∎

Note that in the proof above we required only that $K$ be a reduced ring.

## 5. EXAMPLES AND EXTENSIONS

In this section we describe some of the many examples of lexicographic rings, and suggest some of the ways that our theory may be extended. We have made no attempt to be either complete or exhaustive in our survey, as this field is a very active one and each example is the subject of an entire theory of its own.

EXAMPLE 5.1 (The Chain Transform). This example is thoroughly discussed in Baclawski and Garsia [7]. Let $\Delta$ be a simplicial complex and let $P = P(\Delta)$ be its corresponding poset. Then $K[\Delta]$ is a graded lexicographic ring based on $P$. In [7, Corollary 6.3] it is shown that $K[\Delta]$ is CM if and only if $K[P]$ is CM. We conjecture that one can choose separators for the natural frame of $K[\Delta]$ so that the hypotheses of Theorem 4.7 hold.

EXAMPLE 5.2 (Partition Rings). Let $P$ be a poset. A *P-partition* is an order-reversing function $f\colon P \to \mathbb{N}$; that is, if $x \leqslant y$ in $P$, then $f(x) \geqslant f(y)$. We say $f$ is a *P-partition of n* if $\sum_{x \in P} f(x) = n$. Interpret the elements of $P$ as the indeterminates of the free polynomial ring $K[X] = K[X_p \mid p \in P]$. The *monomial corresponding to* a $P$-partition $f$ is $X^f = \prod_{p \in P} X_p^{f(p)}$. The subalgebra of $K[X]$ generated by the monomials $X^f$ corresponding to the $P$-partitions is called the *partition ring* of $P$. Garsia in [20] explicitly constructs a basic, privileged frame for the partition ring of $P$ by showing that this ring is a graded lexicographic ring based on $J(P)$, the distributive lattice of order-ideals of $P$.

EXAMPLE 5.3 (The Letter-Place Algebra). This ring was introduced by Doubilet *et al.* [17] as a means of developing the invariant theory of the symmetric group in a combinatorial (and characteristic-free) manner. The letter-place algebra is the component of "step zero" of the Cayley algebra (see [17]). More precisely, fix two positive integers $n, m \in \mathbb{N}$. The *letter-place algebra* $R = R(m, n)$ is defined as the free polynomial ring $K[X_{i,j} \mid 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n]$. Define a poset $P(m, n)$ as follows. An element of $P(m, n)$ is an ordered pair of strictly increasing sequences of the same length,

$$(a_1 < a_2 < \cdots < a_k \mid b_1 < b_2 < \cdots < b_k),$$

such that $a_i \in [m]$ and $b_i \in [n]$ for all $i$. The partial order is defined componentwise: $(a_1 < a_2 < \cdots < a_k \mid b_1 < b_2 < \cdots < b_k) \leqslant (c_1 < c_2 < \cdots < c_l \mid d_1 < d_2 < \cdots < d_l)$ if and only if $k \geqslant l$ and for all $i \in [l]$ we have $a_i \leqslant c_i$ and $b_i \leqslant d_i$ ("smaller" in $P(m, n)$ means longer length but smaller entries). The poset $P(m, n)$ is a distributive lattice isomorphic to $J(Q)$, where $Q = [m] \times [n] \backslash \{(m, n)\}$. The *length* of $w = (a_1 < a_2 < \cdots < a_k \mid b_1 < b_2 < \cdots < b_k)$, written $l(w)$, is the integer $k$.

The main result, Theorems 2 and 3, of [17], may be expressed in our language as asserting that the letter-place algebra is a graded, strongly lexicographic ring based on $P(m, n)$. See Stein [37] for generalizations.

Since the letter-place algebra is just a free polynomial ring, there is seemingly little reason for such an elaborate way to describe it. However, the monomial basis given by this description of the letter-place algebra is the only natural basis of $R(m, n)$ with respect to the action of $Gl(m) \times Gl(n)$.

See DeConcini and Procesi [14]. Furthermore, certain quotient rings of the letter-place algebra are also strongly lexicographic. We have in mind two examples, the first is $R/(X_p \mid p \in Q(r))$, where $Q(r) = \{p \in P(m, n) \mid l(p) \geqslant r\}$. This quotient ring is the $r$th determinantal ring of a generic $m \times n$ matrix, i.e., the quotient of the letter-place algebra by all $r \times r$ minors of the matrix $(X_{i,j})$. The other example is the quotient $R/(X_p \mid p \leqslant q)$, for some fixed $q \in P(m, n)$. This quotient ring is related to the coordinate ring of a Schubert variety (see Examples 5.4 and 5.5 below).

An explicit minimal free resolution of the determinantal ring above (in characteristic zero) has been given by Lascoux [29]. The fact that this resolution is based on a form of straightening law analogous to those of Doubilet *et al.* [17] and Désarménien *et al.* [15] leads one to suspect that under suitable conditions the straightening law of a lexicographic ring can be "extended" to straightening laws on its syzygy modules. The results of the next section were largely motivated by this idea.

The poset $P(m, n)$ is a distributive lattice. As a result all of the rings described above are based on semimodular posets. From this one can show that these rings are CM. These rings were originally shown to be CM by Eagon and Hochster [18].

EXAMPLE 5.4 (Bracket Rings). We use the ring $R(m, n)$ defined in Example 5.3, but instead of forming a quotient ring we consider the subalgebra generated by all maximal minors. If we assume that $m \leqslant n$, then these will be the $m \times m$ minors of the matrix $(X_{i,j} \mid 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$. We write $[a_1 < \cdots < a_m]$, where $a_i \in [n]$ for all $i$, for the $m \times m$ minor obtained by using columns $a_1, \ldots, a_m$. The resulting algebra $A(m, n)$ is called a *bracket ring* and is the (homogeneous) coordinate ring of the Grassmannian $\mathrm{Gr}(m-1, n-1)$ of $m$-dimensional subspaces of affine $n$-space, $K^n$. We define a poset $B(m, n)$ as follows. The elements of $B(m, n)$ are the brackets $[a_1 < \cdots < a_m]$, $a_i \in [n]$, and we write $[a_1 < \cdots < a_m] \leqslant [b_1 < \cdots < b_m]$ if and only if $a_i \leqslant b_i$ for every $i$. $B(m, n)$ is isomorphic to $J([m] \times [n-m])$ and hence is CM. The ring $A(m, n)$ is a graded strongly lexicographic ring. This was first shown by Hodge [25] (see also Hodge and Pedoe [26]) who actually considered the more general case of the coordinate ring of a flag manifold. As a result we see that $A(m, n)$ is CM. The fact that $A(m, n)$ is CM was first shown by Laksov [28] and independently by Hochster [23]. Both of these authors used methods different from ours. The first proof using "straightening" methods was found by Musili [30]. The work of Musili played a role in the development of the ideas of DeConcini *et al.* [13].

Bracket rings, in general, are obtained by setting some of the brackets equal to zero. To be more precise, we need some definitions. A *matroid structure* on $[n]$ is a map $f\colon [n] \to S$, where $S$ is the set of elements which cover or are equal to $\hat{0}$ in a geometric lattice $L$. We may assume that the supremum of $f([n])$ is $\hat{1}$. The *bracket ring* of the matroid structure given by $f$ is the quotient ring

$$A(m, n)/([a_1 < \cdots < a_m] \mid f(a_1) \vee \cdots \vee f(a_m) \neq \hat{1}).$$

See Crapo and Rota [10] for the theory of matroids, and see White [39] for the theory of bracket rings. The bracket ring is a universal coordinatizing ring of a given matroid [38, 39].

An important special case of a bracket ring is the homogeneous coordinate ring of a Schubert variety. These are obtained by forming the quotient

$$A(m, n)/([a_1 < \cdots < a_m] \mid [a_1 < \cdots < a_m] \leqslant [b_1 < \cdots < b_m]),$$

for a fixed bracket $[b_1 < \cdots < b_m]$. These are always CM by the same reasoning used in Example 5.3. However, bracket rings, in general, need not be CM. This was shown by White [40].

EXAMPLE 5.5 (Homogenized Rings). The rings in Examples 5.3 and 5.4 are related to one another by a simple process called *homogenization*. The idea is that to an ungraded lexicographic ring $R$ one can associate a graded lexicographic ring $R'$ such that $R$ is obtained from $R'$ by setting one variable in $R'$ equal to a nonzero constant.

Let $R$ be a lexicographic ring based on a poset $P$ of rank $r$, whose admissible order is compatible with some associated grading of $K[X] = K[X_p \mid p \in P]$. We now define a graded lexicographic ring based on $Q$, where $Q = P \cup \{\hat{0}\}$. We give $\hat{0}$ rank 0 so that the rank function on $P$ is the restriction of that on $Q$. The new variable will be denoted $X_0$ and has degree 1. The admissible order is extended to $\mathscr{M}(\{0, 1,..., r\})$ as follows. Since we want this new admissible order to be compatible with degree, we need only consider the set of shapes of a given degree $d$. For each such shape $\lambda$ we have a unique decomposition $\lambda = \lambda_0 + \lambda_1$, where $\lambda_0 \in \mathscr{M}(\{0\})$ and $\lambda_1 \in \mathscr{M}(r)$. We say that $\lambda$ precedes $\lambda'$ if and only if either $|\lambda_0| > |\lambda_1|$ or $|\lambda_0| = |\lambda_1|$ and $\lambda_1$ precedes $\lambda_1'$ in $\mathscr{M}(r)$. It is straightforward to verify that this order is admissible.

We may now define the graded ring $R'$. It is the quotient $K[X_q \mid q \in Q]/I$, where $I$ is generated by all polynomials of the form

$$X_p X_q - h(X_p X_q),$$

where $p, q \in P$ are incomparable and $h$ is defined on monomials of $K[X]$ as follows. First express a monomial $w$ as a linear combination of standard monomials (necessarily of degree of at most that of $w$ in $R$, using the straightening formula). Now multiply each term in this linear combination by a power of $X_0$ so that the resulting term has the same degree as $w$. Then add these up. The result is $h(w)$. We call $h$ the *homogenization function*. Since $\hat{0}$ is comparable to every element of $Q$, $h(w)$ is a linear combination of standard monomials. It is easy to see that $R'$ is a graded lexicographic ring based on $Q$ and that $R \cong R'/(X_0 - 1)$.

As an example, we note that $B(m, n)$ has a maximum element $[n-m+1 < \cdots < n]$ and that $P(m, n-m) \cup \{\hat{0}\} \cong B(m, n)^*$. Now $A(m, n)$ is also a graded lexicographic ring based on $B(m, n)^*$, by dualizing (reversing) the admissible order on shapes of each degree. Furthermore $R(m, n-m) \cong A(m, n)/([n-m+1 < \cdots < n] - 1)$. To see why we consider the matrix

$$\begin{bmatrix} X_{m,1} & \cdots & X_{m,n-m} & 1 & & 0 \\ \cdot & & \cdot & & & \\ \cdot & & \cdot & & 1 & \\ \cdot & & \cdot & & & \ddots \\ X_{1,1} & \cdots & X_{1,n-m} & 0 & & 1 \end{bmatrix}.$$

A maximal minor of this matrix is an arbitrary minor of the submatrix $(X_{i,j} \mid 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n-m)$, the only exception being the minor consisting of the last $m$ columns, $[n-m+1 < \cdots < n]$, is simply the constant 1 (which could be interpreted as the determinant of the minor consisting of no columns and no rows). See DeConcini *et al.* [12] for the proof that these two rings are isomorphic. Note that although $R(m, n-m)$ and $A(m, n)$ are both graded algebras, the isomorphism does not preserve this structure.

*Remark* 5.6 (More General Coefficient Rings). As we remarked in Section 1, the field $K$ may be replaced by a ring, and under suitable hypotheses the results of Section 4 still hold. One may even consider rings, $K$, which are themselves multigraded $K$-algebras. In other words, the elements of $K$ may be sums of elements having nontrivial shape. To be more precise, let $P$ be a poset of rank $r$, and let $K$ be an $\mathbb{N}^r$-graded $K$-algebra. A *lexicographic ring $R$ based on $P$* consists of the following:

(1) a $K$-module isomorphism $\phi: K[P] \to R$,

(2) an admissible order on $\mathcal{M}(r)$ such that if the straightening formula of a nonstandard monomial $X^I$ is $\sum_J \alpha_J X^J$, where each $X^J$ is standard and $\alpha_J \in K$, and if the nonzero homogeneous components of $\alpha_J$ are $\{\alpha_{J,L}\}$, then $\lambda(\alpha_{J,L}) + \lambda(X^J)$ precedes $\lambda(X^I)$ for every $J$ and $L$.

*Remark* 5.7 (Lexicographic Modules). The concept of a lexicographic ring generalizes quite naturally to modules. Suppose that $M$ is a module over $K[a_1, \ldots, a_r]$ and that $P$ is a poset of rank $r$. We say that $M$ is a *lexicographic module* if there is an $\mathbb{N}^r$-graded module $N$ over $K[P]$ and a vector space isomorphism $\phi: N \to M$ such that for any $i$ and homogeneous $m \in N$,

$\phi^{-1}(\phi(\theta_i m) - a_i\phi(m))$ is a linear combination of homogeneous elements of shape preceding $\lambda(\theta_i) + \lambda(m)$. Since the Transfer Theorem and the Cohen–Macaulay property are concerned only with the structure of the ring as a module over a frame, there is no difficulty extending our theory in Section 4 to lexicographic modules.

Examples of such modules have apparently been found by DeConcini [11] who studies the symplectic analogue of the work of Hodge and of Doubilet *et al.* He bases his lexicographic rings on a "Stanley–Reisner ring" over a structure more general than that of a poset. For a more "geometric" approach to the study of these rings (and even more general ones) see LakshmiBai *et al.* [27].

## 6. Betti Numbers

We now show that among all lexicographic rings based on $P$, the Stanley–Reisner ring $K[P]$ has the largest Betti numbers. Some of the results in Section 4 may be given new proofs using this fact. Moreover, we can find new results, involving the canonical module, by using our Betti number bound.

We begin with a technical result. In this theorem we employ the following notation. Suppose that $W$ is a vector space (over $K$) which has a decomposition $W = \bigoplus_{\nu \in Q} W(\nu)$, where $Q$ is a finite poset. An element $w$ of $W$ is said to be *$Q$-homogeneous* if it is in one of the $W(\nu)$ and in this case we say its *shape* is $\nu = \lambda(w)$.

THEOREM 6.1. *Let $\{C_n\}$ be a sequence of finite-dimensional $K$-vector spaces which has two structures of a (algebraic) complex given by maps $\partial_n: C_n \to C_{n-1}$ and maps $\partial'_n: C_n \to C_{n-1}$, respectively. Suppose also that each $C_n$ has a decomposition $C_n = \bigoplus_{\lambda \in Q} C_n(\lambda)$ with respect to some finite poset $Q$. Assume that*

(1) *for every $n$, $\lambda$, $\partial_n(C_n(\lambda)) \subseteq C_{n-1}(\lambda)$,*

(2) *for every $n$, $\lambda$ and $w \in C_n(\lambda)$,*

$$\partial'_n(w) - \partial_n(w) \in \bigoplus_{\nu < \lambda} C_{n-1}(\nu).$$

*Then there is a structure of a complex on $\{H_n(C_*, \partial)\}$ given by maps $\delta_n: H_n \to H_{n-1}$ such that*

$$H_n(H_*, \delta) \cong H_n(C_*, \partial').$$

*Proof.* For any order-ideal $D \subseteq Q$, write $C_n(D)$ for $\bigoplus_{\lambda \in D} C_n(\lambda)$. Hypotheses (1) and (2) ensure that $\partial_n(C_n(D)) \subseteq C_{n-1}(D)$ and $\partial'_n(C_n(D)) \subseteq C_{n-1}(D)$. Moreover, the hypotheses of the theorem also hold with $C_*$ replaced by $C_*(D)$ and $\partial_n, \partial'_n$ replaced by the restrictions $\partial_{D,n}, \partial'_{D,n}$. Thus we may use induction on order-ideals of $Q$.

We will first show by induction that $\operatorname{rank}(\partial'_n) \geqslant \operatorname{rank}(\partial_n)$, for every $n$. (The rank of a linear map is the dimension of its image.) For simplicity we will drop the subscript $n$ in the following argument. Suppose that $\operatorname{rank}(\partial'_D) \geqslant \operatorname{rank}(\partial_D)$ for some order-ideal $D \subseteq Q$. Let $\lambda$ be a minimal element of $Q \backslash D$. We will show that $\operatorname{rank}(\partial'_E) \geqslant \operatorname{rank}(\partial_E)$, where $E = D \cup \{\lambda\}$.

To this end we define $s_\lambda$ to be the map $s_\lambda = \partial'_\lambda - \partial_\lambda\colon C_n(\lambda) \to C_{n-1}(D)$. This map is well defined by hypothesis (2). Next define the subspace $U(\lambda) = \operatorname{Ker}(\partial_\lambda) \cap s_\lambda^{-1}(\operatorname{Im}(\partial'_D))$. We now choose a basis $\{w_i\}$ for $U(\lambda)$. By definition, each such element is in $s_\lambda^{-1}(\operatorname{Im}(\partial'_D))$, i.e., $s_\lambda(w_i) \in \operatorname{Im}(\partial'_D)$. Choose an element $y_i \in C_n(E)$ such that $s_\lambda(w_i) = \partial'_D(y_i)$ for each $i$. We then use these choices to define a linear map $h\colon U(\lambda) \to C_n(D)$ by $h(\sum a_i w_i) = \sum a_i y_i$. By linearity, $s_\lambda(w) = \partial'_D(h(w))$ for every $w \in U(\lambda)$. Finally, we define a map $\phi\colon U(\lambda) \oplus \operatorname{Ker}(\partial'_D) \to C_n(D)$ by $\phi(w+x) = w - h(w) + x$ for $w \in U(\lambda)$ and $x \in \operatorname{Ker}(\partial'_D)$.

We now check that $\operatorname{Im}(\phi) \subseteq \operatorname{Ker}(\partial'_E)$. To this end, we compute $\partial'_E(\phi(w+x)) = \partial'(w - h(w) + x) = \partial'_\lambda(w) + \partial'_D(x - h(w)) = \partial'_\lambda(w) + \partial'_D(x) - \partial'_D(h(w)) = \partial_\lambda(w) + s_\lambda(w) + \partial'_D(x) - s_\lambda(w) = \partial_\lambda(w) + \partial'_D(x) = 0$, since $w \in U(\lambda) \subseteq \operatorname{Ker}(\partial_\lambda)$ and $x \in \operatorname{Ker}(\partial'_D)$.

Next we show that $\operatorname{Ker}(\partial'_E) = \operatorname{Im}(\phi)$. Let $w + x$ be in $\operatorname{Ker}(\partial'_E)$, where $w \in C_n(\lambda)$ and $x \in C_n(D)$. Now $\partial'_E(w+x) = \partial'_\lambda(w) + \partial'_D(x) = \partial_\lambda(w) + s_\lambda(w) + \partial'_D(x)$. Since $\partial'_D(x) + s_\lambda(x) \in C_{n-1}(D)$, the element $w + x$ can be in $\operatorname{Ker}(\partial'_E)$ if and only if $\partial_\lambda(w) = 0$ and $-s_\lambda(w) = \partial'_D(x)$. In particular, for such an element we have that $w \in \operatorname{Ker}(\partial_\lambda) \cap s_\lambda^{-1}(\operatorname{Im}(\partial'_D)) = U(\lambda)$. Thus $h(w)$ is defined. It is easy to check that $x + h(w) \in \operatorname{Ker}(\partial'_D)$ and that $\phi(w + x + h(w)) = w + x$. Thus $\operatorname{Ker}(\partial'_E) = \operatorname{Im}(\phi)$ as desired. Since $\phi$ is obviously injective, we then have that $\dim_K(\operatorname{Ker}(\partial'_E)) = \dim_K(U(\lambda)) + \dim_K(\operatorname{Ker}(\partial'_D))$.

Let $c_D = \dim_K(C_n(D))$. Similarly define $c_E$ and $c_\lambda$. We know by the inductive hypothesis that $\operatorname{rank}(\partial'_D) \geqslant \operatorname{rank}(\partial_D)$. Then

$$
\begin{aligned}
\operatorname{rank}(\partial'_E) &= c_E - \dim_K(\operatorname{Ker}(\partial'_E)) \\
&= c_E - \dim_K(U(\lambda)) - \dim_K(\operatorname{Ker}(\partial'_D)) \\
&= c_\lambda - \dim_K(U(\lambda)) + c_D - \dim_K(\operatorname{Ker}(\partial'_D)) \\
&\geqslant c_\lambda - \dim_K(\operatorname{Ker}(\partial_\lambda)) + c_D - \dim_K(\operatorname{Ker}(\partial'_D)) \\
&= \operatorname{rank}(\partial_\lambda) + \operatorname{rank}(\partial'_D) \\
&\geqslant \operatorname{rank}(\partial_\lambda) + \operatorname{rank}(\partial_D) \\
&= \operatorname{rank}(\partial_E).
\end{aligned}
$$

The last equality is a consequence of the fact that $\partial_E = \partial_\lambda \oplus \partial_D$. Thus by induction, $\operatorname{rank}(\partial'_n) \geqslant \operatorname{rank}(\partial_n)$ for every $n$.

In fact, we see that the above discussion may be used to compute $\operatorname{rank}(\partial'_n)$. One simply arranges the elements of $Q$ in a sequence $\lambda_1, \ldots, \lambda_q$ in such a way that $\lambda_i < \lambda_j \Rightarrow i < j$. Next define $U_i = U(\lambda_i)$ using the order-ideal $D = \{\lambda_1, \ldots, \lambda_{i-1}\}$ as we did above. Then

$$\operatorname{rank}(\partial'_n) = \dim_K(C_n) - \sum_{i=1}^{q} \dim_K(U_i).$$

We are now in a position to prove the theorem. Choose splittings $C_n = B_n \oplus E_n \oplus D_n = B'_n \oplus E'_n \oplus D'_n$ so that $B_n = \operatorname{Im}(\partial_{n+1})$, $B'_n = \operatorname{Im}(\partial'_{n+1})$, $B_n \oplus E_n = \operatorname{Ker}(\partial_n)$, $B'_n \oplus E'_n = \operatorname{Ker}(\partial'_n)$, $\partial_n: D_n \simeq B_{n-1}$, and $\partial'_n: D'_n \simeq B'_{n-1}$. Note that $H_n(C_*, \partial) \cong E_n$, $H_n(C_*, \partial') \cong E'_n$, $\dim(B_n) \leqslant \dim(B'_n)$, $\dim(E_n) \geqslant \dim(E'_n)$ and $\dim(D_n) \leqslant \dim(D'_n)$. Now choose injective maps $\beta_n: D_n \to D'_n$ and splittings $D'_n = \beta_n(D_n) \oplus D''_n$. Since $\partial'_n: D'_n \simeq B'_{n-1}$, $\dim(\partial'_n(\beta_n(D_n))) = \dim(D_n) = \dim(B_{n-1})$. Thus we may choose an injective map $\alpha_n: B_n \to B'_n$ so that $\alpha_n(B_n) = \partial'_{n+1}(\beta_{n+1}(D_{n+1}))$. Furthermore, the splitting $D'_{n+1} = \beta_{n+1}(D_{n+1}) \oplus D''_{n+1}$ may be transferred to $B_n$ via $\partial'_{n+1}$, so that $B'_n = \alpha_n(B_n) \oplus B''_n$, where $B''_n$ is defined as $\partial'_{n+1}(D''_{n+1})$. It is easy to check that $\dim(E_n) = \dim(B''_n \oplus E'_n \oplus D''_n)$. Thus we may choose isomorphisms $\gamma_n: E_n \simeq B''_n \oplus E'_n \oplus D''_n$. Note that since $B_n = \partial_{n+1}(D_{n+1})$, we have $\alpha_n(\partial_{n+1}(D_{n+1})) = \partial'_{n+1}(\beta_{n+1}(D_{n+1}))$. We may combine $\alpha_n$, $\beta_n$ and $\gamma_n$ to get an automorphism

$$\begin{aligned} \omega_n: C_n = B_n \oplus E_n \oplus D_n &\simeq \alpha_n(B_n) \oplus B''_n \oplus E'_n \oplus D''_n \oplus \beta_n(D_n) \\ &= B'_n \oplus E'_n \oplus D'_n \\ &= C_n. \end{aligned}$$

Finally, we define $\delta_n: C_n \to C_{n-1}$ by $\delta_n = \omega_{n-1}^{-1} \circ \partial'_n \circ \omega_n$, so that this diagram commutes:

$$\begin{tikzcd} C_n \arrow[r, "\omega_n"] \arrow[d, "\delta_n"'] & C_n \arrow[d, "\partial'_n"] \\ C_{n-1} \arrow[r, "\omega_{n-1}"] & C_{n-1}. \end{tikzcd}$$

Since every $\omega_n$ is an automorphism, $(C_*, \delta)$ is a complex isomorphic to $(C_*, \partial')$.

We now show that for every $n$ we have the inclusions

$$\operatorname{Im}(\partial_{n+1}) \subseteq \operatorname{Im}(\delta_{n+1}) \subseteq \operatorname{Ker}(\delta_n) \subseteq \operatorname{Ker}(\partial_n). \qquad (*)$$

To this end, let $b \in B_n = \mathrm{Im}(\partial_{n+1})$. Then $\omega_n(b) = \alpha_n(b) \in B'_n = \mathrm{Im}(\partial'_{n+1})$. Thus there is some $a \in C_{n+1}$ such that $\partial'_{n+1}(a) = \omega_n(b)$. We then compute that $\delta_{n+1}(\omega_{n+1}^{-1}(a)) = \omega_n^{-1}(\partial'_{n+1}(a)) = \omega_n^{-1}(\omega_n(b)) = b$. Hence $b \in \mathrm{Im}(\delta_{n+1})$. This gives the first inclusion of (∗). The third inclusion of (∗) is given by a similar calculation, and the second inclusion is a consequence of the fact that $\delta_n \circ \delta_{n+1} = 0$.

Now the inclusions in (∗) immediately imply that $\delta_n(\mathrm{Ker}(\partial_n)) \subseteq \mathrm{Im}(\delta_n) \subseteq \mathrm{Ker}(\partial_{n-1})$ and that $\delta_n(\mathrm{Im}(\partial_{n+1})) \subseteq \delta_n(\mathrm{Ker}(\delta_n)) = (0) \subseteq \mathrm{Im}(\partial_n)$. Thus for every $n$, $\delta_n$ induces a map

$$\bar{\delta}_n: H_n(C_*, \partial) \to H_{n-1}(C_*, \partial).$$

This sequence of maps obviously defines a structure of a complex on $\{H_n(C_*, \partial)\}$. The map $\partial'_n$ restricts to a map

$$\partial''_n: B''_n \oplus E'_n \oplus D''_n \to B''_{n-1} \oplus E'_{n-1} \oplus D''_{n-1},$$

because $\partial'_n D''_n = B''_{n-1}$. The construction of the automorphism $\omega_n$ then ensures that

$$\gamma_{n-1}^{-1} \circ \partial''_n \circ \gamma_n: E_n \to E_{n-1}$$

is the restriction of $\delta_n$. Thus we have a commutative diagram

$$\begin{tikzcd}
B''_n \oplus E'_n \oplus D''_n \arrow[r, "\partial''_n"] & B''_{n-1} \oplus E'_{n-1} \oplus D''_{n-1} \\
H_n(C_*, \partial) \arrow[u, "\wr"'] \arrow[r, "\bar{\delta}_n"] & H_{n-1}(C_*, \partial), \arrow[u, "\wr"']
\end{tikzcd}$$

where the vertical isomorphisms are induced by $\gamma_n$ and $\gamma_{n-1}$, respectively. Since we have $\partial''_n(D''_n) = B''_{n-1}$ and $\mathrm{Ker}(\partial''_n) = B''_n \oplus E'_n$, it follows that

$$H_n(H_*(C_*, \partial), \bar{\delta}_n) \cong E'_n.$$

Finally, by the choice of $E'_n$, we have $E'_n \cong H_n(C_*, \partial')$. The theorem then follows. ∎

We now come to the main result of this section, which asserts that the *graded* Betti numbers of a graded lexicographic ring over $P$ are bounded by the corresponding Betti numbers of the Stanley–Reisner ring $K[P]$. Moreover, they are bounded in a special manner to be described below that restricts the possible sets of Betti numbers even more.

THEOREM 6.2. *Let $R$ be a graded lexicographic ring based on $P$. Then for every integer $m$ there is a structure of a complex on $\mathscr{H}_m \mathrm{Tor}_*^{K[X]}(K[P], K)$,* *whose* $n$th *homology is isomorphic to* $\mathscr{H}_m \operatorname{Tor}_n^{K[X]}(R, K)$. *In particular, for every* $m$ *and* $n$, *we have* $b_{n,m}(R) \leqslant b_{n,m}(K[P])$; *and for every* $n$, *we have* $b_n(R) \leqslant b_n(K[P])$.

*Proof.* We begin by describing the Koszul complex of $\{X_p \mid p \in P\}$ over $K[X]$. This complex is a minimal finite free resolution of $K$ over $K[X]$. The $n$th free module $F_n$ in the Koszul complex has a basis $\{e(S) \mid S \subseteq P, |S| = n\}$ over $K[X]$. Choose a total order on the elements of $P$. If $S \subseteq P$ and $p \in P$, we write $m(p, S)$ for the number of elements of $S$ that strictly precede $p$ in the chosen total order on $P$. The structure map $\partial_n: F_n \to F_{n-1}$ of the Koszul complex may be expressed as

$$\partial_n(e(S)) = \sum_{p \in S} (-1)^{m(p,S)} X_p e(S \backslash \{p\}).$$

We define the degree of $e(S)$ as $\deg(e(S)) = \sum_{p \in S} \deg(X_p)$; with this definition, the $F_n$ become graded modules over $K[X]$, and the maps $\partial_n$ are homogeneous degree-preserving maps.

We now tensor the Koszul complex with $K[P]$ and with $R$. Now as graded vector spaces, $F_n \otimes_{K[X]} K[P]$ and $F_n \otimes_{K[X]} R$ are identical. Thus we are in the situation of Theorem 6.1, i.e., we have two structures of a complex on $C_* = F_* \otimes K[P]$. These complexes may be described as follows. A basis element of $C_n$ (over $K$) has the form $X^I e(S)$, where $I \in \mathscr{M}(\Delta(P))$, $S \subseteq P$ and $|S| = n$. We define the *shape* of $X^I e(S)$ to be $\lambda(X^I e(S)) = \lambda(X^I) + \sum_{p \in S} \lambda(X_p)$. The structure map $\partial_n$ for $F_* \otimes K[P]$ is given by

$$\partial_n(X^I e(S)) = \sum_{\substack{p \in S \\ \square(I) \cup \{p\} \in \Delta(P)}} (-1)^{m(p,S)} X^I X_p e(S \backslash \{p\}).$$

Clearly, $\partial_n$ preserves shapes. The structure map $\partial'_n$ for $F_* \otimes R$ is the same as $\partial_n$ except that when $\square(I) \cup \{p\} \notin \Delta(P)$ we replace $X^I X_p$ by its straightening formula. Each term appearing in this linear combination has a shape that strictly precedes $\lambda(X^I) + \lambda(X_p)$. Multiplying this straightening formula by $e(S \backslash \{p\})$ simply adds the shape $\lambda(e(S \backslash \{p\}))$ to every term. By the admissibility of the partial order on shapes, we conclude that $(\partial'_n - \partial_n)(X^I e(S))$ is a linear combination of terms whose shapes strictly precede $\lambda(X^I e(S))$.

Thus the hypotheses of Theorem 6.1 are satisfied except for the finiteness of $Q$ and the finite dimensionality of $C_n$. However, we *can* apply Theorem 6.1 to each homogeneous component of $C_*$ and then combine these, since all the structure maps are homogeneous and preserve degree. Since $\mathscr{H}_m \operatorname{Tor}_n^{K[X]}(K[P], K) = H_n(\mathscr{H}_m C_*, \partial)$ and $\mathscr{H}_m \operatorname{Tor}_n^{K[X]}(R, K) = H_n(\mathscr{H}_m C_*, \partial')$, the result follows. ∎

An immediate consequence of Theorem 6.2 is that Corollary 4.6 holds without the requirement that the poset $P$ be ACM. We also obtain another proof of DeConcini's conjecture.

Because of Theorem 6.2, it is of interest to know the graded Betti numbers of a Stanley–Reisner ring. To compute these, we need to introduce the (topological) cohomology of a simplicial complex. Let $\Delta$ be a simplicial complex on a vertex set $V$. Choose a total order on $V$. We define a cocomplex $(C^*, \delta)$ as follows: $C^n$ is the vector subspace of $K[\Delta]$ spanned by $\{X^I \mid I \in \Delta,\ |I| = n+1\}$ and $\delta^n\colon C^n \to C^{n+1}$ is the linear map defined by $\delta^n(X^I) = \sum_{v \in V\setminus I, I \cup \{v\} \in \Delta} (-1)^{m(v,I)} X^I X_v$, where $m(v, I)$ is the number of elements of $I$ that strictly precede $v$. The $n$th reduced cohomology of $\Delta$ is then defined to be the $n$th cohomology of $(C^*, \delta)$. This is not the usual way that one defines $\tilde{H}^n(\Delta, K)$, but it is easy to see that it is equivalent to the standard approach. The dimension of $\tilde{H}^n(\Delta, K)$ is written $\tilde{h}_n(\Delta, K)$ and is called the $n$th *reduced Betti number of $\Delta$ over $K$*.

We need one more bit of notation. If we give $K[P]$ one of its associated gradings, we write $\deg(S)$ for $\deg(X^S)$, where $S$ is any subset of $P$. We now state our computation.

PROPOSITION 6.3 (Hochster). *Let $P$ be a finite poset. Then*

$$b_{n,m}(K[P]) = \sum_{\substack{S \subseteq P \\ \deg(S) = m}} \tilde{h}_{|S|-n-1}(\Delta(S), K),$$

$$b_n(K[P]) = \sum_{S \subseteq P} \tilde{h}_{|S|-n-1}(\Delta(S), K).$$

Actually, Hochster [24] only computed the second formula above, but did so for any simplicial complex, not just one of the form $\Delta(P)$.

*Proof.* We use the complex $(C_*, \partial)$ defined in the proof of Theorem 6.2 to compute $\mathscr{H}_m \operatorname{Tor}_n^{K[X]}(K[P], K)$. The structure map of this complex preserves more than just shape, it preserves the *content* of a basis element $X^I e(S)$, where content$(X^I e(S))$ is the multiset $I + S$.

Now the restriction of the Koszul complex to a particular content $T \in \mathscr{M}(P)$ may be expressed as follows. The structure map is given on basis elements by $\partial_n(X^I e(S)) = \sum_{p \in S, \square(I) \cup \{p\} \in \Delta(P)} (-1)^{m(p,S)} X^I X_p e(S \setminus \{p\})$, where $n = |S|$. We now define $A = \square(T)$ and $B = T \setminus A$ so that $T$ is the multiset union of $A$ and $B$. The elements of $B$ are the repeated elements of the multiset $T$. Let $C = \square(B)$. Now in the term $X^I e(S)$, $S$ is a *set* and $T = I + S$. Thus $B$ is a submultiset of $I$, i.e., $X^I = X^J X^B$ for some $J \subseteq A$. Note that $J$ has no multiplicities since $B$ already accounts for these. Finally, we observe that since the content $T$ is fixed, once we know $J = I \setminus B$, we can reconstruct the basis element $X^I e(S)$. Thus if we "factor out" $X^B$ and suppress $e(S)$'s, the map $\partial_n$ map be expressed as

$$\partial_n(X^J) = \sum_{\substack{p \in A \backslash J \\ C \cup J \cup \{p\} \in \Delta(A)}} (-1)^{m(p, A \backslash J)} X^J X_p,$$

where $n = |A \backslash J|$. The condition $C \cup J \cup \{p\} \in \Delta(A)$ is equivalent to $J \cup \{p\} \in \operatorname{star}_{\Delta(A)}(C) = \{D \mid D \cup C \in \Delta(A)\}$. Looking at the definition of the reduced cohomology of $\operatorname{star}_{\Delta(A)}(C)$, we see that $\mathscr{H}_T \operatorname{Tor}_n^{K[X]}(K[P], K) \cong \tilde{H}^{|A|-n-1}(\operatorname{star}_{\Delta(A)}(C), K)$, where "$\mathscr{H}_T$" means the homogeneous part of content $T$. Now $\operatorname{star}_{\Delta(A)}(C)$ is acyclic if $C \neq \varnothing$. Thus we may assume that the content $T$ is a set. In this case $A = T$ so that $\mathscr{H}_T \operatorname{Tor}_n^{K[X]}(K[P], K) \cong \tilde{H}^{|T|-n-1}(\Delta(T), K)$. The proposition now follows since $\mathscr{H}_m = \bigoplus_{\deg(T)=m} \mathscr{H}_T$. ∎

## 7. Canonical Modules and Gorenstein Rings

Let $R$ be a CM graded $K$-algebra. If we dualize a minimal free resolution of $R$, the result will be a minimal free resolution of a graded module over $R$ called the *canonical module* of $R$, written $\Omega(R)$. Although one must use a presentation of $R$ as a quotient of a free polynomial ring in order to compute $\Omega(R)$, up to a shift in degree $\Omega(R)$ depends only on $R$. Thus any information about $\Omega(R)$ is a property of $R$ alone (unlike the Betti numbers, for example). One such concept is the *type* of $R$ which is defined to be the multiset of degrees of a minimal set of generators of $\Omega(R)$ as a module over $R$. Our definition differs slightly from the standard one, which we would denote by $|\operatorname{type}(R)|$, i.e., the minimum number of generators of $\Omega(R)$ over $R$. The ring $R$ is said to be *Gorenstein* if $|\operatorname{type}(R)| = 1$. In this case one can show that $R$ is isomorphic to $\Omega(R)$ by a homogeneous map of degree $m$, where $\{m\} = \operatorname{type}(R)$ depends on the presentation of $R$. See Herzog and Kunz [21]. Now it is an easy consequence of Theorem 6.2 that if $R$ is a graded lexicographic ring based on a Gorenstein poset, then $R$ is also Gorenstein. We will show that the converse also holds in certain circumstances, but first we must introduce a new concept.

Let $P$ be a finite poset, and let $k > 0$ be an integer. We say that $P$ is *$k$-Cohen–Macaulay connected* (or simply $k$-CM) if for every $S \subseteq P$ such that $|S| < k$, we have

(1) $P \backslash S$ is CM, and

(2) $r(P \backslash S) = r(P)$.

This concept was introduced by the author in [4]. Among other examples, it was shown there that if $\hat{P}$ is a geometric lattice, then $P$ is doubly CM;

moreover, for such posets one can give a simple characterization of the $k$-CM property for $k > 2$. Thus examples of $k$-CM posets for any $k$ are readily found. Our main result connecting the theory of doubly CM posets with rings is the following theorem. To state this result, we need to recall that the *reduced Euler characteristic*, written $\mu(P)$, of a poset is the alternating sum $\sum_{n=-1}^{\infty}(-1)^n \tilde{h}_n(\Delta(P), K)$.

THEOREM 7.1. *Let $P$ be a 2-CM poset, and let $R$ be a graded lexicographic ring based on $P$. Then as multisets* $\operatorname{type}(R) = \operatorname{type}(K[P])$. *Moreover, in this case* $|\operatorname{type}(R)| = |\mu(P)|$. *In particular, $R$ is Gorenstein if and only if $\mu(P) = \pm 1$.*

*Proof.* Let $r = r(P)$, $n = |P| - r$ and $d = \deg(P)$. By [4, Theorem 4.5] and Proposition 6.3, if $P$ is 2-CM, then

$$\begin{aligned} b_{n,m}(K[P]) &= (-1)^{r-1}\mu(P), \qquad && \text{if} \quad m = d \\ &= 0, && \text{otherwise.} \end{aligned}$$

Thus there is only one nonzero homogeneous component of $\operatorname{Tor}_n^{K[X]}(K[P], K)$, and it has degree $d$. Moreover since $P$ is the only subset of $P$ having degree $d$, and since $P$ is CM, we also have that $b_{p,d}(K[P]) = 0$ for every $p \neq n$. We now apply Theorem 6.2 to conclude that $b_{n,d}(R) = b_{n,d}(K[P]) = (-1)^{r-1}\mu(P)$ and that $b_{n,m}(R) = b_{n,m}(K[P]) = 0$ for $m \neq d$. We conclude that $\operatorname{Tor}_n^{K[X]}(K[P], K)$ and $\operatorname{Tor}_n^{K[X]}(R, K)$ are isomorphic as graded vector spaces. Thus $\operatorname{type}(R) = \operatorname{type}(K[P])$ and $|\operatorname{type}(R)| = |\mu(P)|$. The rest of the theorem now follows from these equations and our earlier remarks. ∎

One can produce further refinements of the result in Theorem 7.1 by assuming that $P$ is $k$-CM for some $k > 2$. We will consider just the next case in the following theorem.

THEOREM 7.2. *Let $R$ be a graded lexicographic ring based on a 3-CM poset $P$. Then*

(1) *the minimum number of generators of $\Omega(R)$ is $|\mu(R)|$,*

(2) *the minimum number of relations in the presentation of $\Omega(R)$ over $K[X]$ is $\sum_{x \in P} |\mu(P \backslash \{x\})|$.*

We might add that the degree of each generator of $\Omega(R)$ is $(-\deg(P))$ while the $|\mu(P\backslash\{x\})|$ relations supplied by the subset $P\backslash\{x\}$ are each of degree $(-\deg(P\backslash\{x\}))$. Also if we make use of the incidence algebra $I(\hat{P}, \mathbb{Q})$ of $\hat{P}$ as introduced by Rota [32], then we may also express the minimum number of relations as $(|P| + 2)\,|\mu(P)| + |(\mu * \mu)(\hat{0}, \hat{1})|$.

*Proof.* Part (1), of course, just follows from Theorem 7.1. The second part follows from the case $k = 3$ of [4, Theorem 4.5] by exactly the same proof as in Theorem 7.1 above. ▮

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